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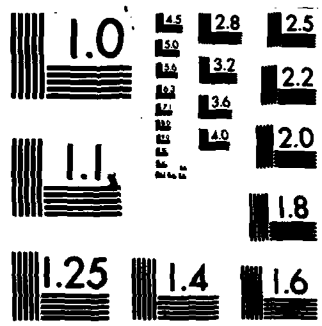
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CONVEXITY AND CONCAVITY PROPERTIES OF THE OPTIMAL VALUE FUNCTION IN PARAMETRIC NONLINEAR PROGRAMMING

by

Anthony V. Fiacco
Jerzy Kyparisis

Program in Logistics
GWU/IMSE/Serial-T-471/82
21 December 1982

THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
Institute for Management Science and Engineering
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Convexity and concavity properties of the optimal value function f^* are considered for the general parametric optimization problem $P(\epsilon)$ of the form $\min_x f(x,\epsilon)$ s.t. $x \in R(\epsilon)$. Such properties of f^* and the solution set map S^* form an important part of the theoretical basis for sensitivity, stability, and parametric analysis in mathematical optimization. Sufficient conditions are given for several standard types of convexity and concavity

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20. Abstract (cont'd)

of f^* , in terms of respective convexity and concavity assumptions on f and the feasible region point-to-set map R . Specializations of these results to the general parametric inequality-equality constrained nonlinear programming problem and its right-hand-side version are provided. Convexity properties of the solution point-to-set map S^* for the general problem $P(\epsilon)$ are also briefly considered. Although most of the results appear to be new, some basic results were obtained previously in a somewhat different setting. These are included here and related to the new developments, thus providing the first comprehensive survey of these important characterizations.

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1. INTRODUCTION

We consider a general parametric optimization problem of the form

$$\min_x \; f(x,\varepsilon) \quad \text{s.t.} \quad x \in R(\varepsilon) \qquad P(\varepsilon)$$

where $f\colon E^n \times E^k \to E^1$ and R is a point-to-set map from E^k to E^n, as well as several specializations of this problem. Our primary interest is the characterization of convexity and concavity properties of the optimal value function f^* of the problem $P(\varepsilon)$ (sometimes called the perturbation function or the marginal function), defined as

$$f^*(\varepsilon) = \begin{cases} \inf_x \; \{f(x,\varepsilon) \mid x \in R(\varepsilon)\}, & \text{if } R(\varepsilon) \neq \phi \\ +\infty, & \text{if } R(\varepsilon) = \phi \end{cases}$$

Sufficient conditions for several types of convexity and concavity of f^*, such as standard, strict, polyhedral, uniform, and homogeneous, are given in terms of assumptions on f and the feasible set map R.

Surprisingly, most of the results seem to be new, despite the common character of these notions. Convexity properties of the solution point-to-set map S^* defined by

$$S^*(\varepsilon) = \{x \in R(\varepsilon) \mid f(x,\varepsilon) = f^*(\varepsilon)\}$$

are also briefly considered.

Convexity, concavity, and other fundamental properties of the optimal value function f^* and the solution set map S^*, such as continuity, differentiability, and so forth, form a theoretical basis for sensitivity, stability, and parametric analysis in nonlinear optimization. Consequently, interest in such properties and their applications has always been manifest [e.g., see Berge (1963) and Rockafellar (1970)]. Developments in this general area have been intensive during the past decade and there have been considerable current efforts to provide a unified body of theory and methodology. This is evidenced in particular by the recent emergence of several monographs, by Rockafellar (1974), Brosowski (1982), Bank, et al. (1982), and Fiacco (1983).

However, despite these important advances and the fact that a few results concerning convexity and concavity of f^* have been known for some time, there does not appear to be any comprehensive or systematic treatment of these properties in the literature. This paper endeavors to provide this much-needed treatment, collecting important known results and presenting many new results in a unified manner. Readers less familiar with the various (generalized) notions of convexity and concavity utilized here may consult the books by Mangasarian (1969) and Ortega and Rheinboldt (1970), or a recent survey by Avriel, et al. (1981) and the references contained therein.

We now specify several programs studied in this paper. Along with problem $P(\varepsilon)$, we shall also consider a less general parametric optimization problem,

$$\min_{x} \ f(x) \quad \text{s.t.} \quad x \in R(\varepsilon) \qquad P'(\varepsilon)$$

By specializing the feasible set (constraint set) map R of $P(\varepsilon)$, we obtain the parametric nonlinear programming (NLP) problem,

$$\min_{x \in M} \ f(x,\varepsilon) \quad \text{s.t.} \quad \begin{array}{ll} g_i(x,\varepsilon) \geqslant 0 , & i = 1,\ldots,m \\ h_j(x,\varepsilon) = 0 , & j = 1,\ldots,p \end{array} \qquad P_3(\varepsilon)$$

where $M \subset E^n$, $g_i: E^n \times E^k \to E^1$, $i = 1,\ldots,m$, $h_j: E^n \times E^k \to E^1$, $j = 1,\ldots,p$, i.e., with R defined by

$$R(\varepsilon) = \{x \in M \mid g_i(x,\varepsilon) \geqslant 0,\ i = 1,\ldots,m,\ h_j(x,\varepsilon) = 0,\ j = 1,\ldots,p\} .$$

We may further specialize $P_3(\varepsilon)$ to the general right-hand-side (grhs) NLP problem,

$$\min_{x \in M} \ f(x,\varepsilon) \quad \text{s.t.} \quad \begin{array}{ll} g_i(x) \geqslant \varepsilon_i , & i = 1,\ldots,m \\ h_j(x) = \varepsilon_{m+j} , & j = 1,\ldots,p \end{array} \qquad P_2(\varepsilon)$$

i.e., with R defined by

$$R(\varepsilon) = \{x \in M \mid g_i(x) \geqslant \varepsilon_i,\ i = 1,\ldots,m,\ h_j(x) = \varepsilon_{m+j},\ j = 1,\ldots,p\} .$$

Note that $P_2(\varepsilon)$ differs from the standard rhs NLP problem for which $f(x,\varepsilon) = f(x)$. Finally, we also consider a problem of the form

$$\min_{x} \ f(x,\varepsilon) \quad \text{s.t.} \quad x \in M \qquad P_1(\varepsilon)$$

In the sequel several notions from convex analysis and the theory of point-to-set maps will be frequently used. We define them here for completeness. The set $M \subset E^r$ is a convex set if for any $x_1,x_2 \in M$ and $\lambda \in [0,1]$, $\lambda x_1 + (1-\lambda)x_2 \in M$. The convex hull of a set $A \subset E^r$

is the set $\mathrm{conv}(A) = \{\lambda x_1 + (1-\lambda)x_2 \mid x_1, x_2 \in A, \lambda \in [0,1]\}$. The set $K \subset E^r$ is a cone if $x \in K$ implies $\lambda x \in K$ for all $\lambda > 0$, and K is a convex cone if it is also convex, i.e., if $\lambda x_1 + \mu x_2 \in K$ for all $x_1, x_2 \in K$ and $\lambda, \mu > 0$. A point-to-set map $R: E^k \to E^n$ assigns a subset of E^n (possibly empty) to each element of E^k . The domain of R is the set $\mathrm{dom}R = \{\varepsilon \in E^k \mid R(\varepsilon) \neq \phi\}$. The graph G of R is defined by $G(R) = \{(\varepsilon, x) \mid x \in R(\varepsilon)\}$. Also, we define the range of R over A by $R(A) = \bigcup_{\varepsilon \in A} R(\varepsilon)$ for any set $A \subset E^k$.

For these and other notions of linear algebra and convex analysis, and topology and point-to-set map theory, the reader is referred to the books by Rockafellar (1970) and Berge (1963), respectively.

Several immediate extensions and applications of the results given here are noted. Firstly, most of the results remain valid in more general spaces, e.g., real vector, real vector topological, or Banach spaces. Secondly, the results for $P(\varepsilon)$ and $P'(\varepsilon)$ are applicable to ri- ous problems of abstract optimal control theory, mathematical economics, etc. Thirdly, many specific problems of nonlinear programming, e.g., geometric programming, separable programming, fractional programming, etc. often possess structures that enable one to apply many of the results given here. Finally, it is possible to obtain many other results on generalized convexity and concavity properties of the optimal value function f^* . This will be the topic of the forthcoming paper.

Applications of some of the results given here to the construction of simple upper and lower bounds on f^* in parametric nonlinear and geometric programming have been given in Fiacco (1983) and Kyparisis (1982). (A first such application was probably the well known sensitivity result of Everett (1963) for rhs convex programming problems.)

A number of related recent results are not studied here, e.g., interesting results on paraconvex point-to-set maps [see Rolewicz (1979, 1980,1981)] and results concerning (global) Lipschitz continuity of f^* [see, e.g., Stern and Topkis (1976), Dolecki (1977,1978), and Robinson (1981)]. Also, some extensions to multiobjective optimization were obtained by Tanino and Sawaragi (1979,1980).

2. CONVEXITY OF THE OPTIMAL VALUE FUNCTION

An (extended) function $\phi: E^r \to E^1 \cup \{-\infty,\infty\}$ is called "convex" on a convex set $M \subset E^r$ [see, e.g., Rockafellar (1970)], if the epigraph set of f on M, $\{(x,r) \mid x \in M,\ r \geq f(x)\}$ is convex, or equivalently if for all $x_1, x_2 \in M$ and $\lambda \in (0,1)$,

$$\phi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\phi(x_1) + (1-\lambda)\phi(x_2)$$

(where we adopt the convention that $(-\infty) + \infty = \infty$). The following notion is well known.

The point-to-set map $R: E^k \to E^n$ is called "convex" on a convex set $S \subset E^k$ if the set $G(R) \cap (S \times E^n)$ is convex, or equivalently if, for all $\varepsilon_1, \varepsilon_2 \in S$ and $\lambda \in (0,1)$,

$$\lambda R(\varepsilon_1) + (1-\lambda)R(\varepsilon_2) \subset R(\lambda\varepsilon + (1-\lambda)\varepsilon_2) .$$

[For a more general notion of a "B-convex" point-to-set map, see Borwein (1977) and Tanino and Sawaragi (1979).] We introduce a slight extension of the previous notion and call $R: E^k \to E^n$ "essentially convex" on a convex set $S \subset E^k$ if, for all $\varepsilon_1, \varepsilon_2 \in S$, $\varepsilon_1 \neq \varepsilon_2$, and $\lambda \in (0,1)$,

$$\lambda R(\varepsilon_1) + (1-\lambda)R(\varepsilon_2) \subset R(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2) .$$

It is clear that if R is convex on S , then it is essentially convex on S . A convex map R on S is also convex-valued on S , i.e., its "value" $R(\varepsilon)$ at $\varepsilon \in S$ is a convex set. However, an essentially convex map may not be convex-valued at the boundary points of S , as shown below.

Example 2.1

Consider $R: E^2 \to E^1$ defined by

$$R(\varepsilon_1,\varepsilon_2) = \begin{cases} [0,1] , & \text{if } \varepsilon_1^2 + \varepsilon_2^2 < 1 \\ \{0\} \cup \{1\} , & \text{if } \varepsilon_1^2 + \varepsilon_2^2 = 1 \\ \phi , & \text{if } \varepsilon_1^2 + \varepsilon_2^2 > 1 \end{cases}$$

and $S = \{(\varepsilon_1,\varepsilon_2) \mid \varepsilon_1^2 + \varepsilon_2^2 \leq 1\}$. If is easy to check that R is essentially convex on S but $R(\varepsilon_1,\varepsilon_2)$ is not convex if $\varepsilon_1^2 + \varepsilon_2^2 = 1$.

Proposition 2.2

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly convex on the set $\{(x,\varepsilon) \mid x \in R(\varepsilon), \varepsilon \in S\}$, R is essentially convex or convex on S and S is convex, then f^* is convex on S .

Proof: Let $\varepsilon_1,\varepsilon_2 \in S$, $\varepsilon_1 \neq \varepsilon_2$, and $\lambda \in (0,1)$. Then by convexity of f and essential convexity of R , we obtain

$$\begin{aligned} f^*\big(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) &= \inf_{x \in R(\lambda\varepsilon_1+(1-\lambda)\varepsilon_2)} f\big(x,\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) \\ &\leq \inf_{\substack{x_1 \in R(\varepsilon_1)\\ x_2 \in R(\varepsilon_2)}} f\big(\lambda x_1 + (1-\lambda)x_2, \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) \\ &\leq \inf_{\substack{x_1 \in R(\varepsilon_1)\\ x_2 \in R(\varepsilon_2)}} [\lambda f(x_1,\varepsilon_1) + (1-\lambda)f(x_2,\varepsilon_2)] \end{aligned}$$

$$= \lambda \inf_{x_1 \varepsilon R(\varepsilon_1)} f(x_1,\varepsilon_1) + (1-\lambda) \inf_{x_2 \varepsilon R(\varepsilon_2)} f(x_2,\varepsilon_2)$$

$$= \lambda f^*(\varepsilon_1) + (1-\lambda) f^*(\varepsilon_2) \ ,$$

i.e., f^* is convex on S . ■

The above result is quite well known when R is convex and often appears in a different but essentially equivalent form: if $f\colon E^n \times E^k \to E^1 \cup \{-\infty,\infty\}$ is a jointly convex function (also called a bifunction), then f^* given by $f^*(\varepsilon) = \inf_x\{f(x,\varepsilon) \mid x \varepsilon E^n\}$ is convex [Rockafellar (1970, Thm. 29.1; 1974)]. If we assume that $f > -\infty$ on $E^n \times E^k$ and define the point-to-set map R_0 by $R_0(\varepsilon) = \{x \varepsilon E^n \mid f(x,\varepsilon) < +\infty\}$, then the problems

$$\min_x \ f(x,\varepsilon) \quad \text{s.t.} \quad x \varepsilon E^n \qquad P_0(\varepsilon)$$

and

$$\min_x \ f(x,\varepsilon) \quad \text{s.t.} \quad x \varepsilon R_0(\varepsilon) \qquad P(\varepsilon)$$

are equivalent. We prefer the latter form of the general problem since it allows us to treat the constraints explicitly. Also, in the case of essentially convex R , an equivalent problem $P_0(\varepsilon)$ will not necessarily have jointly convex $\tilde{f}$, where we define

$$\tilde{f}(x,\varepsilon) = \begin{cases} f(x,\varepsilon) \ , & \text{if} \quad (\varepsilon,x) \ \varepsilon \ G(R) \\ +\infty \ , & \text{if} \quad (\varepsilon,x) \notin G(R) \ . \end{cases}$$

Another extension of the notion of a convex point-to-set map is introduced next. We call $R\colon E^k \to E^n$ a "closure convex" point-to-set map on a convex set $S \subset E^k$ if, for all $\varepsilon_1,\varepsilon_2 \ \varepsilon \ S$ and $\lambda \ \varepsilon \ (0,1)$,

$$\lambda R(\varepsilon_1) + (1-\lambda)R(\varepsilon_2) \subset c\ell\big(R(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2)\big)$$

where $c\ell(A)$ denotes the topological closure of the set A. We observe that R is closure convex on S if and only if the map $c\ell R$ given by $c\ell R(\varepsilon) = c\ell\big(R(\varepsilon)\big)$ is convex on S (this follows from the fact that $c\ell(A) + c\ell(B) \subset c\ell(A+B)$ for arbitrary sets A and B). Also, if R is convex on S, then it is closure convex on S, but not conversely, as shown in the next example.

<u>Example 2.3</u>

Consider $R: E^1 \to E^1$ defined by

$$R(\varepsilon) = \begin{cases} (0,1) , & \text{if } |\varepsilon| < 1 \\ [0,1] , & \text{if } |\varepsilon| = 1 \\ \emptyset , & \text{if } |\varepsilon| > 1 \end{cases}$$

and $S = \{\varepsilon \mid |\varepsilon| \leq 1\}$. It is easily seen that R is closure convex but not convex on S (note that R is convex-valued on S). Similarly, we call R "essentially closure convex" on a convex set S if $c\ell R$ is essentially convex on S.

<u>Proposition 2.4</u>

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly convex on the set $\{(x,\varepsilon) \mid x \in c\ell\big(R(\varepsilon)\big), \varepsilon \in S\}$ and upper semicontinuous in x on the set $c\ell\big(R(S)\big)$ for every $\varepsilon \in S$, R is essentially closure convex or closure convex on S and S is convex, then f^* is convex on S.

<u>Proof</u>: Define $\bar{R}(\varepsilon) = c\ell\big(R(\varepsilon)\big)$ for $\varepsilon \in S$ and denote $\bar{f}^*(\varepsilon) = \inf_x\{f(x,\varepsilon) \mid x \in \bar{R}(\varepsilon)\}$. By Proposition 2.2 $\bar{f}^*$ is convex on S, since $\bar{R}$ is essentially convex on S. We now show that $\bar{f}^*(\varepsilon) = f^*(\varepsilon)$ for $\varepsilon \in S$. Obviously, $f^*(\varepsilon) \geq \bar{f}^*(\varepsilon)$ for $\varepsilon \in S$. Let $\{x_n\} \subset \bar{R}(\varepsilon)$, $\varepsilon \in S$, be such that $\bar{f}^*(\varepsilon) = \lim_{n\to\infty} f(x_n,\varepsilon)$ (such a sequence exists by

definition of infimum). Then for every n there exists a sequence $\{x_{nm}\} \subset R(\varepsilon)$ such that $x_n = \lim_{m\to\infty} x_{nm}$. Thus by upper semicontinuity of f ,

$$f(x_n,\varepsilon) = f\left(\lim_{m\to\infty} x_{nm}, \varepsilon\right) \geq \limsup_{m\to\infty} f(x_{nm},\varepsilon) \geq f^*(\varepsilon) ,$$

which implies that $\bar{f}^*(\varepsilon) \geq f^*(\varepsilon)$, $\varepsilon \in S$. ■

<u>Remark 2.5</u>

In the above proof we have shown that if f is upper semicontinuous in x and $R(\varepsilon)$ is an arbitrary set, then

$$\bar{f}^*(\varepsilon) = \inf_{x\in c\ell(R(\varepsilon))} f(x,\varepsilon) = \inf_{x\in R(\varepsilon)} f(x,\varepsilon) = f^*(\varepsilon) .$$

In the following we consider specializations of Proposition 2.2 to problems $P_3(\varepsilon)$ and $P_2(\varepsilon)$ [several other specializations are given in Rockafellar (1974)]. In order to state sufficient conditions for the convexity of R and f^* for the parametric NLP problem $P_3(\varepsilon)$, we need the following definitions. A function ϕ: $E^r \to E^1 \cup \{-\infty,\infty\}$ is called "quasiconvex" on a convex set $M \subset E^r$ [Fenchel (1953)] if the level sets $L_c = \{x \in M \mid \phi(x) \leq c\}$ are convex for all $c \in E^1 \cup \{-\infty,\infty\}$, or equivalently, if for all $x_1,x_2 \in M$ and $\lambda \in (0,1)$,

$$\phi\left(\lambda x_1 + (1-\lambda)x_2\right) \leq \max\{\phi(x_1),\phi(x_2)\} .$$

A convex function on a convex set M is also quasiconvex on M .

A function ϕ: $E^r \to E^1 \cup \{-\infty,\infty\}$ is called "quasiconcave" on a convex set $M \subset E^r$ [Fenchel (1953)] if $-\phi$ is quasiconvex on M , i.e., if the level sets $U_c = \{x \in M \mid \phi(x) \geq c\}$ are convex for all $c \in E^1 \cup \{-\infty,\infty\}$.

The function ϕ is called "quasimonotonic" on a convex set $M \subset E^r$ [Martos (1967)] if ϕ is both quasiconvex and quasiconcave on M, i.e., if the sets $E_c = \{x \in M \mid \phi(x) = c\}$ are convex for all $c \in E^1 \cup \{-\infty,\infty\}$. If ϕ is affine on M, then clearly it is also quasimonotonic on M.

Proposition 2.6

Consider the parametric NLP problem $P_3(\varepsilon)$. If $\{g_i\}$ are jointly quasiconcave on $M \times S$, $\{h_j\}$ are jointly quasimonotonic on $M \times S$ and M and S are convex sets, then R given by $R(\varepsilon) = \{x \in M \mid g_i(x,\varepsilon) \geq 0,\ i = 1,\ldots,m,\ h_j(x,\varepsilon) = 0,\ j = 1,\ldots,p\}$ is convex on S.

Proof: Consider

$$G(R) \cap (S \times E^n) = \{(\varepsilon,x) \in S \times M \mid g_i(x,\varepsilon) \geq 0,\ i = 1,\ldots,m\}$$
$$\cap \{(\varepsilon,x) \in S \times M \mid h_j(x,\varepsilon) = 0,\ j = 1,\ldots,p\} .$$

By quasiconcavity of $\{g_i\}$ and quasimonotonicity of $\{h_j\}$ on $M \times S$ this set is convex, implying convexity of R on S. ■

The following result is now immediate.

Corollary 2.7

Consider the parametric NLP problem $P_3(\varepsilon)$. If f is jointly convex on $M \times S$, $\{g_i\}$ are jointly quasiconcave on $M \times S$, $\{h_j\}$ are jointly quasimonotonic on $M \times S$, and M and S are convex, then f^* is convex on S.

Proof: This follows directly from Propositions 2.2 and 2.6. ■

Corollary 2.7 generalizes the result of Mangasarian and Rosen (1964), who assumed joint concavity (see the definition following) of $\{g_i\}$ to obtain convexity of f^* (for the problem with inequality constraints only).

A function $\phi: E^r \to E^1 \cup \{-\infty,\infty\}$ is called "concave" on a convex set $M \subset E^r$ [see, e.g., Rockafellar (1970)] if $-\phi$ is convex on M (see also Section 3 for this definition). It is called "affine" on M if it is finite and both convex and concave on M (see also Section 4 for this definition).

<u>Proposition 2.8</u>

Consider the grhs parametric NLP problem $P_2(\varepsilon)$. If $\{g_i\}$ are concave on M, $\{h_j\}$ are affine on M and M is convex, then R given by $R(\varepsilon) = \{x \in M \mid g_i(x) \geq \varepsilon_i,\ i = 1,\ldots,m,\ h_j(x) = \varepsilon_{m+j},\ j = 1,\ldots,p\}$ is convex.

<u>Proof</u>: This result follows directly from Proposition 2.6, since $\tilde{g}_i(x,\varepsilon) = g_i(x) - \varepsilon_i$ is jointly quasiconcave and $\tilde{h}_j(x,\varepsilon) = h_j(x) - \varepsilon_{m+j}$ is jointly quasimonotonic on $M \times E^k$ $(k = m+p)$ under the above assumptions. ■

Note that quasiconcavity of $\{g_i\}$ in Proposition 2.8 is not sufficient for the result to hold. To see this, consider the problem $\min\ x$, s.t. $e^x \geq \varepsilon$ $(x \in E^1,\ \varepsilon > 0)$. Obviously, e^x is quasiconcave but $f^*(\varepsilon) = \ell n(\varepsilon)$ is not convex.

<u>Corollary 2.9</u>

Consider the grhs parametric NLP problem $P_2(\varepsilon)$. If f is jointly convex on $M \times S$, $\{g_i\}$ are concave on M, $\{h_j\}$ are affine on M, and M and S are convex, then f^* is convex on S.

Proof: Follows immediately from Propositions 2.2 and 2.8. ∎

The above result is very well known for the standard rhs NLP problem, i.e., when $f(x,\varepsilon) = f(x)$ [see, e.g., Rockafellar (1970, Thm. 29.1), Luenberger (1969, §8.3), and Geoffrion (1971)]. Note that for $P_1(\varepsilon)$, joint convexity of f on $M \times S$ and convexity of M and S imply convexity of f^* on S (since if M is convex, then R defined by $R(\varepsilon) = M$, $\varepsilon \in S$, is convex on S).

We introduce still another extension of the notion of a convex point-to-set map next. We call $R: E^k \to E^n$ a "hull convex" ("essentially hull convex") point-to-set map on a convex set $S \subset E^k$ if the map convR given by $\mathrm{conv}R(\varepsilon) = \mathrm{conv}(R(\varepsilon))$ is convex (essentially convex) on S, or equivalently, if for all $\varepsilon_1, \varepsilon_2 \in S$ ($\varepsilon_1 \neq \varepsilon_2$) and $\lambda \in (0,1)$,

$$\lambda \mathrm{conv}\big(R(\varepsilon_1)\big) + (1-\lambda)\mathrm{conv}\big(R(\varepsilon_2)\big) \subset \mathrm{conv}\big(R(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2)\big) .$$

Since for all $\varepsilon_1, \varepsilon_2 \in S$ and $\lambda \in (0,1)$,

$$\lambda \mathrm{conv}\big(R(\varepsilon_1)\big) + (1-\lambda)\mathrm{conv}\big(R(\varepsilon_2)\big) = \mathrm{conv}\big(\lambda R(\varepsilon_1) + (1-\lambda)R(\varepsilon_2)\big) ,$$

we can use the condition

$$\lambda R(\varepsilon_1) + (1-\lambda)R(\varepsilon_2) \subset \mathrm{conv}\big(R(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2)\big)$$

in the above definition. If R is convex on a convex set S, then R is hull convex on S but not vice versa. For example, if M is not convex, then R given by $R(\varepsilon) = M$, $\varepsilon \in S$, is hull convex but not convex on S. These two notions coincide if R is convex-valued; thus a hull convex R is not necessarily convex-valued. The next result partially generalizes Proposition 2.2 (see also Proposition 3.4 and Remark 3.5).

Proposition 2.10

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly convex on the set $\{(x,\varepsilon) \mid x \in R(\varepsilon), \varepsilon \in S\}$ and quasiconcave in x on E^n for every $\varepsilon \in S$, R is essentially hull convex or hull convex on S, and S is a convex set, then f^* is convex on S.

Proof: Define $\tilde{R}(\varepsilon) = \text{conv}\big(R(\varepsilon)\big)$ for $\varepsilon \in S$. Denote $\tilde{f}^*(\varepsilon) = \inf_x\{f(x,\varepsilon) \mid x \in \tilde{R}(\varepsilon)\}$. By Proposition 2.2 $\tilde{f}^*$ is convex on S, since $\tilde{R}$ is convex on S. We show that actually $f^*(\varepsilon) = \tilde{f}^*(\varepsilon)$ for $\varepsilon \in S$. Clearly, $f^*(\varepsilon) \geqslant \tilde{f}^*(\varepsilon)$, $\varepsilon \in S$. Suppose that $\varepsilon \in S$ and $x \in \tilde{R}(\varepsilon)$. Then $x = \lambda x_1 + (1-\lambda)x_2$ for some $x_1, x_2 \in R(\varepsilon)$ and $\lambda \in (0,1)$. By quasiconcavity of f

$$f(x,\varepsilon) = f\big(\lambda x_1 + (1-\lambda)x_2,\varepsilon\big) \geqslant \min\{f(x_1,\varepsilon),\ f(x_2,\varepsilon)\}$$

$$\geqslant \inf_{x \in R(\varepsilon)} f(x,\varepsilon) = f^*(\varepsilon) .$$

This shows that $\tilde{f}^*(\varepsilon) \geqslant f^*(\varepsilon)$. ■

The notion of hull convexity can also be used to extend the definitions of polyhedral and homogeneous convex maps and maps convex at a point (all of which are given later in this section) and the corresponding results concerning f^* can be similarly partially extended.

We shall now restrict the usual notions of convexity as follows. We call a set $M \subset E^r$ "convex at $x_0 \in M$" (also called "star-shaped at x_0") if, for any $x \in M$ and $\lambda \in (0,1)$, $\lambda x_0 + (1-\lambda)x \in M$. A function $\phi: E^r \to E^1 \cup \{-\infty,\infty\}$ is called "convex at x_0" ("concave at x_0") on a set $M \subset E^r$ convex at x_0 if, for all $x \in M$ and $\lambda \in (0,1)$,

$$\phi\big(\lambda x_0 + (1-\lambda)x\big) \leqslant (\geqslant)\ \lambda\phi(x_0) + (1-\lambda)\phi(x)$$

[see Mangasarian (1969) for a slightly more general definition, without assuming convexity of M at x_0].

We call ϕ "affine at x_0" on M convex at x_0 if it is finite and both convex and concave at x_0 on M. We now extend the notion of a convex map and call $R: E^k \to E^n$ "convex at ε_0" on a set S convex at ε_0 if $G(R) \cap (S \times E^n)$ is convex at (x,ε_0) for all $x \in R(\varepsilon_0)$, or equivalently, if for all $\varepsilon \in S$ and $\lambda \in (0,1)$,

$$\lambda R(\varepsilon_0) + (1-\lambda)R(\varepsilon) \subset R\big(\lambda\varepsilon_0 + (1-\lambda)\varepsilon\big) .$$

Furthermore, we call R "essentially convex at ε_0" on S convex at ε_0 if the above inclusion holds for all $\varepsilon \in S$, $\varepsilon \neq \varepsilon_0$ and $\lambda \in (0,1)$. This latter definition is slightly more general and allows for nonconvexity of $R(\varepsilon_0)$. We state two results whose proofs parallel those of Proposition 2.2 and *Corollary 2.7 and are therefore not given.*

Proposition 2.11

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly convex at (x,ε_0) for all $x \in R(\varepsilon_0)$ on $E^n \times S$, R is essentially convex or convex at ε_0 on S and S is convex at ε_0, then f^* is convex at ε_0 on S.

Proposition 2.12

Consider the parametric NLP problem $P_3(\varepsilon)$. If f is jointly convex at (x,ε_0) for all $x \in R(\varepsilon_0)$ on $M \times S$, $\{g_i\}$ are jointly concave at (x,ε_0) for all $x \in R(\varepsilon_0)$ on $M \times S$, $\{h_j\}$ are jointly affine at (x,ε_0) for all $x \in R(\varepsilon_0)$ on $M \times S$, M is convex and S is convex at ε_0, then f^* is convex at ε_0 on S.

A function $\phi: E^r \to E^1 \cup \{-\infty,\infty\}$ is called "strictly convex" on a convex set $M \subset E^r$ [see, e.g., Rockafellar (1970)] if, for all $x_1, x_2 \in M$, $x_1 \neq x_2$ and $\lambda \in (0,1)$,

$$\phi\left(\lambda x_1 + (1-\lambda)x_2\right) < \lambda\phi(x_1) + (1-\lambda)\phi(x_2) .$$

Proposition 2.13

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly strictly convex on the set $\{(x,\varepsilon) \mid x \in R(\varepsilon),\ \varepsilon \in S\}$, R is essentially convex or convex on S, S is convex and $S^*(\varepsilon) \neq \phi$ for all $\varepsilon \in S$ (i.e., there exist solutions to $P(\varepsilon)$ for all $\varepsilon \in S$), then f^* is strictly convex on S.

Proof: Let $\varepsilon_1, \varepsilon_2 \in S$, $\varepsilon_1 \neq \varepsilon_2$, and $\lambda \in (0,1)$. Denote $\varepsilon_\lambda = \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2$. By our assumptions there exist $x_1^* \in S^*(\varepsilon_1)$ and $x_2^* \in S^*(\varepsilon_2)$, so by strict convexity of f and essential convexity or convexity of R we obtain

$$\begin{aligned}
\lambda f^*(\varepsilon_1) + (1-\lambda)f^*(\varepsilon_2) &= \lambda f(x_1^*,\varepsilon_1) + (1-\lambda)f(x_2^*,\varepsilon_2) \\
&> f\left(\lambda x_1^* + (1-\lambda)x_2^*, \varepsilon_\lambda\right) \\
&\geq \inf_{\substack{x_1 \in R(\varepsilon_1) \\ x_2 \in R(\varepsilon_2)}} f\left(\lambda x_1 + (1-\lambda)x_2, \varepsilon_\lambda\right) \\
&\geq \inf_{x \in R(\varepsilon_\lambda)} f(x,\varepsilon_\lambda) = f^*(\varepsilon_\lambda) ,
\end{aligned}$$

i.e., f^* is strictly convex on S. ■

For the NLP problem $P_3(\varepsilon)$ we obtain the following result. A similar result can be stated for $P_2(\varepsilon)$.

Proposition 2.14

Consider the parametric NLP problem $P_3(\varepsilon)$. If f is jointly strictly convex on $M \times S$, $\{g_i\}$ are jointly quasiconcave on $M \times S$, $\{h_j\}$ are jointly quasimonotonic on $M \times S$, M and S are convex, and $S^*(\varepsilon) \neq \phi$ for all $\varepsilon \in S$, then f^* is strictly convex on S.

Proof: Follows from Propositions 2.13 and 2.6 combined. ■

Proposition 2.13 is not applicable if $f(x,\varepsilon) = f(x)$ for the problem $P'(\varepsilon)$, but we can state another result that does apply to this problem.

Proposition 2.15

Consider the parametric optimization problem $P'(\varepsilon)$. If f is strictly convex on the set $R(S)$, R is convex on S, S is convex, $S^*(\varepsilon) \neq \phi$ for all $\varepsilon \in S$, and $S^*(\varepsilon_1) \neq S^*(\varepsilon_2)$ if $\varepsilon_1,\varepsilon_2 \in S$, $\varepsilon_1 \neq \varepsilon_2$, then f^* is strictly convex on S.

Proof: Let $\varepsilon_1,\varepsilon_2 \in S$, $\varepsilon_1 \neq \varepsilon_2$ and $\lambda \in (0,1)$. By our assumptions there exist $x_1^* \in S^*(\varepsilon_1)$, $x_2^* \in S^*(\varepsilon_2)$, and $x_1^* \neq x_2^*$. Thus, by strict convexity of f and convexity of R we have

$$\begin{aligned}\lambda f^*(\varepsilon_1) + (1-\lambda)f^*(\varepsilon_2) &= \lambda f(x_1^*) + (1-\lambda)f(x_2^*) \\ &> f\big(\lambda x_1^* + (1-\lambda)x_2^*\big) \geq \inf_{\substack{x_1\in R(\varepsilon_1)\\ x_2\in R(\varepsilon_2)}} f\big(\lambda x_1 + (1-\lambda)x_2\big) \\ &\geq \inf_{x\in R(\lambda\varepsilon_1+(1-\lambda)\varepsilon_2)} f(x) = f^*\big(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) ,\end{aligned}$$

i.e., f^* is strictly convex on S. ■

Remark 2.16

Note that in Propositions 2.13 - 2.15, the $S^*(\varepsilon)$ are actually singleton sets. Hence, the assumption that $S^*(\varepsilon_1) \neq S^*(\varepsilon_2)$ in Proposition 2.15 means that S^* (treated as a single-valued map) is one-to-one on S .

Specialization of this result to the NLP problem $P_3(\varepsilon)$ can be easily obtained using Proposition 2.6.

Next, we consider the following strengthened notion of convexity. We call $\phi: E^r \to E^1 \cup \{-\infty,\infty\}$ "uniformly convex with $d(\cdot)$ " on a convex set $M \subset E^r$ [see Ortega and Rheinboldt (1970) for a slightly different definition] if, for all $x_1, x_2 \in M$ and $\lambda \in (0,1)$,

$$\phi\big(\lambda x_1 + (1-\lambda)x_2\big) \leq \lambda\phi(x_1) + (1-\lambda)\phi(x_2) - \lambda(1-\lambda)d(\|x_1 - x_2\|) ,$$

where $d: [0,\infty) \to [0,\infty)$ is an increasing function, with $d(t) > 0$ for $t > 0$ and $d(0) = 0$, and $\|\cdot\|$ is an arbitrary norm in E^r .

Proposition 2.17

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly uniformly convex with $d(\cdot)$ on the set $\{(x,\varepsilon) \mid x \in R(\varepsilon), \varepsilon \in S\}$, R is essentially convex or convex on S and S is a convex set, then f^* is uniformly convex with $d(\cdot)$ on S .

Proof: Let $\varepsilon_1, \varepsilon_2 \in S$, $\varepsilon_1 \neq \varepsilon_2$ and $\lambda \in (0,1)$. By our assumptions and the properties of the function d and the norm, we obtain

$$f^*\big(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) = \inf_{x \in R(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2)} f\big(x, \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big)$$

$$\leq \inf_{\substack{x_1 \in R(\varepsilon_1) \\ x_2 \in R(\varepsilon_1)}} f\big(\lambda x_1 + (1-\lambda)x_2, \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big)$$

$$\leq \inf_{\substack{x_1 \varepsilon R(\varepsilon_1) \\ x_2 \varepsilon R(\varepsilon_2)}} [\lambda f(x_1,\varepsilon_1) + (1-\lambda) f(x_2,\varepsilon_2) - \lambda(1-\lambda) d(\|(x_1,\varepsilon_1) - (x_2,\varepsilon_2)\|)]$$

$$= \inf_{\substack{x_1 \varepsilon R(\varepsilon_1) \\ x_2 \varepsilon R(\varepsilon_2)}} [\lambda f(x_1,\varepsilon_1) + (1-\lambda) f(x_2,\varepsilon_2) - \lambda(1-\lambda) d(\|(x_1-x_2,\ \varepsilon_1-\varepsilon_2)\|)]$$

$$\leq \inf_{\substack{x_1 \varepsilon R(\varepsilon_1) \\ x_2 \varepsilon R(\varepsilon_2)}} [\lambda f(x_1,\varepsilon_1) + (1-\lambda) f(x_2,\varepsilon_2)] - \lambda(1-\lambda) d(\|\varepsilon_1-\varepsilon_2\|)$$

$$= \lambda \inf_{x_1 \varepsilon R(\varepsilon_1)} f(x_1,\varepsilon_1) + (1-\lambda) \inf_{x_2 \varepsilon R(\varepsilon_2)} f(x_2,\varepsilon_2) - \lambda(1-\lambda) d(\|\varepsilon_1-\varepsilon_2\|)$$

$$= \lambda f^*(\varepsilon_1) + (1-\lambda) f^*(\varepsilon_2) - \lambda(1-\lambda) d(\|\varepsilon_1-\varepsilon_2\|)$$

i.e., f^* is uniformly convex with $d(\cdot)$ on S . ■

Remark 2.18

Note that actually for the above result to hold, we need only assume that, for any $x_1, x_2 \varepsilon E^n$, $\varepsilon_1, \varepsilon_2 \varepsilon S$ and $\lambda \varepsilon (0,1)$,

$$f(\lambda x_1 + (1-\lambda)x_2,\ \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2) \leq \lambda f(x_1,\varepsilon_1) + (1-\lambda) f(x_2,\varepsilon_2) - \lambda(1-\lambda) d(\|\varepsilon_1-\varepsilon_2\|) .$$

This could be termed uniform convexity of f in ε with joint convexity in (x,ε) .

Another observation is that Proposition 2.17, for the parametric optimization problem $P'(\varepsilon)$, reduces to Proposition 2.2, i.e., one obtains only convexity of f^* . Finally, in the standard definition of uniform convexity, one sets $d(t) = \gamma t^2$, $\gamma > 0$ [Pol'yak (1966), see also Avriel, et al. (1981)] and the term "strong convexity" is used in this case.

Specializations of the above result to the parametric NLP problem $P_3(\varepsilon)$ and the grhs parametric NLP problem $P_2(\varepsilon)$ can be easily stated in a manner similar to Corollaries 2.7 and 2.10. Moreover, one can define the notion of uniform convexity at a point and prove a result similar to that in Proposition 2.11.

A "convex polyhedron" in E^r is any set of the form $\{x \in E^r \mid Ax \leqslant b\}$, where A is a $s \times r$ matrix and $b \in E^s$. A function $\phi: E^r \to E^1 \cup \{-\infty,\infty\}$ is called "polyhedral convex" on a convex polyhedron $M \subset E^r$ [see, e.g., Rockafellar (1970)] if the epigraph set $\{(x,r) \mid x \in M,\ r \geqslant \phi(x)\}$ is a convex polyhedron. We call $R: E^k \to E^n$ a "polyhedral convex" point-to-set map on a convex polyhedron $S \subset E^k$ [see, e.g., Rockafellar (1967) for a special case of this notion] if the set $G(R) \cap (S \times E^n)$ is a convex polyhedron. A more general notion of a "polyhedral" point-to-set map was given by Robinson (1981) (see Section 4).

Proposition 2.19

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly polyhedral convex on the set $\{(x,\varepsilon) \mid x \in R(\varepsilon),\ \varepsilon \in S\}$, R is polyhedral convex on S, and S is a convex polyhedron, then f^* is polyhedral convex on S.

Proof: Define $\tilde{f}: E^n \times E^k \to E^1 \cup \{\infty\}$ by

$$\tilde{f}(x,\varepsilon) = \begin{cases} f(x,\varepsilon), & \text{if } x \in R(\varepsilon) \text{ and } \varepsilon \in S \\ +\infty, & \text{otherwise.} \end{cases}$$

By our assumptions $\tilde{f}$ is polyhedral convex on $E^n \times E^r$. Thus, by Rockafellar (1970, Thm. 29.2), $f^*(\varepsilon) = \inf_x\{\tilde{f}(x,\varepsilon) \mid x \in E^n\}$ is polyhedral convex on S. ∎

A function $\phi: E^r \to E^1 \cup \{-\infty,\infty\}$ is called "polyhedral concave" on a convex polyhedron $M \subset E^r$ if $-\phi$ is polyhedral convex on M. Proposition 2.19 can now be specialized to the problem $P_3(\varepsilon)$ as follows.

Proposition 2.20

Consider the parametric NLP problem $P_3(\varepsilon)$. If f is jointly polyhedral convex on $M \times S$, $\{g_i\}$ are jointly polyhedral concave on $M \times S$, $\{h_j\}$ are jointly affine on $M \times S$, and M and S are convex polyhedra, then f^* is polyhedral convex on S.

Proof: Consider $G(R) \cap (S \times E^n) = \{(\varepsilon,x) \in S \times M \mid g_i(x,\varepsilon) \geq 0, i = 1,\ldots,m\} \cap \{(\varepsilon,x) \in S \times M \mid h_j(x,\varepsilon) = 0, j = 1,\ldots,m\}$. By our assumptions this set is a convex polyhedron, hence R is polyhedral convex on S. The result now follows from Proposition 2.19. ■

A similar result can also be given for the grhs parametric NLP program $P_2(\varepsilon)$ using Proposition 2.8.

Finally, we consider the notion of homogeneity and homogeneous convexity. We call $\phi: E^r \to E^1 \cup \{-\infty,\infty\}$ a "positively homogeneous" function on a cone $K \subset E^r$ [see, e.g., Rockafellar (1970)] if $\phi(\lambda x) = \lambda\phi(x)$ for all $x \in K$ and $\lambda > 0$. We call $R: E^k \to E^n$ a "positively homogeneous" point-to-set map on a cone $K \subset E^k$ [see, e.g., Rockafellar (1967)] if $R(\lambda\varepsilon) = \lambda R(\varepsilon)$ for all $\varepsilon \in K$ and $\lambda > 0$, or equivalently, if $G(R) \cap (K \times E^n)$ is a cone. Note that if ϕ is positively homogeneous on K and $0 \in K$, then $\phi(0) = 0$ if $\phi(0)$ is finite. Also, if R is positively homogeneous on K and $0 \in K$, then $0 \in R(0)$ if $R(0)$ is a nonempty closed set.

Proposition 2.21

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly positively homogeneous on $E^n \times K$, R is positively homogeneous on K, and K is a cone, then f^* is positively homogeneous on K.

Proof: Let $\lambda > 0$ and $\varepsilon \in K$. Then $\lambda\varepsilon \in K$ and by our assumptions,

$$f^*(\lambda\varepsilon) = \inf_{x \in R(\lambda\varepsilon)} f(x,\lambda\varepsilon) = \inf_{x \in R(\varepsilon)} f(\lambda x,\lambda\varepsilon) = \lambda \inf_{x \in R(\varepsilon)} f(x,\varepsilon)$$

$$= \lambda f^*(\varepsilon) . \qquad \blacksquare$$

The following specialization to $P_3(\varepsilon)$ is immediate.

Proposition 2.22

Consider the parametric NLP problem $P_3(\varepsilon)$. If f, $\{g_i\}$ and $\{h_j\}$ are jointly positively homogeneous on $M \times K$, and M and K are cones, then f^* is positively homogeneous on K.

Proof: In view of Proposition 2.21 we need only prove that R is positively homogeneous on K, i.e., that $G(R) \cap (K \times E^n)$ is a cone. Let $(\varepsilon,x) \in G(R) \cap (K \times E^n)$ and $\lambda > 0$ be arbitrary. Then $g_i(x,\varepsilon) \geq 0$, $i = 1,\ldots,m$, $h_j(x,\varepsilon) = 0$, $j = 1,\ldots,p$, where $x \in M$, $\varepsilon \in K$. By the assumptions above, $\lambda x \in M$, $\lambda\varepsilon \in K$, and $g_i(\lambda x,\lambda\varepsilon) = \lambda g_i(x,\varepsilon) \geq 0$, $i = 1,\ldots,m$, $h_j(\lambda x,\lambda\varepsilon) = \lambda h_j(x,\varepsilon) = 0$, $j = 1,\ldots,p$, implying that $\lambda(\varepsilon,x) \in G(R) \cap (K \times E^n)$. $\blacksquare$

Further specialization to $P_2(\varepsilon)$ is obvious. The next, more special result applies in particular to $P_1(\varepsilon)$.

Proposition 2.23

Consider again the general problem $P(\varepsilon)$. If f is positively homogeneous in ε on K for every $x \in E^n$, $R(\lambda\varepsilon) = R(\varepsilon)$ for any $\varepsilon \in K$ and $\lambda > 0$, and K is a cone, then f^* is positively homogeneous on K.

Proof: Let $\varepsilon \in K$ and $\lambda > 0$. Then $\lambda\varepsilon \in K$ and we obtain

$$f^*(\lambda\varepsilon) = \inf_{x \in R(\lambda\varepsilon)} f(x,\lambda\varepsilon) = \inf_{x \in R(\varepsilon)} \lambda f(x,\varepsilon) = f^*(\varepsilon) . \quad \blacksquare$$

Similar results to Propositions 2.21 - 2.23 were obtained by Borwein (1980) for the problem

$$\min_{x \in E^n} f(x,\varepsilon) \qquad P_0(\varepsilon)$$

where $f\colon E^n \times E^k \to E^1 \cup \{-\infty,\infty\}$.

In accordance with the previous definitions, a function $\phi\colon E^r \to E^1 \cup \{-\infty,\infty\}$ is called "homogeneous convex (concave)" on a convex cone $K \subset E^r$ if it is both positively homogeneous on K and convex (concave) on K [see, e.g., Rockafellar (1970)]. As is well known, if $\phi\colon E^r \to E^1$ is both homogeneous convex and homogeneous concave on a convex cone K, then ϕ is linear on K. A point-to-set map $R\colon E^k \to E^n$ will be called "homogeneous convex" on a convex cone $K \subset E^k$ if it is both convex and positively homogeneous on K, or equivalently, if $G(R) \cap (K \times E^n)$ is a convex cone. Rockafellar (1967,1970) introduced and extensively studied such maps under the name of "convex processes" [he requires that $(0,0) \in G(R) \cap (K \times E^n)$]. He also considered subclasses of convex processes called "monotone processes of a convex (concave) type."

Proposition 2.24

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly homogeneous convex on $E^n \times K$, R is homogeneous convex on K, and K is a convex cone, then f^* is homogeneous convex on K.

Proof: Follows directly from Propositions 2.2 and 2.21 taken together.■

We call $\phi: E^r \to E^1 \cup \{-\infty,\infty\}$ "homogeneous quasiconvex (quasiconcave)" on a convex cone $K \subset E^r$ if ϕ is both positively homogeneous and quasiconvex (quasiconcave) on K. A specialization of the above conclusions yields the next result.

Proposition 2.25

Consider the general parametric NLP problem $P_3(\varepsilon)$. If f is jointly homogeneous convex on $M \times K$, $\{g_i\}$ are jointly homogeneous quasiconcave on $M \times k$, $\{h_j\}$ are jointly linear on $M \times K$, and M and K are convex cones, then f^* is homogeneous convex on K.

Proof: Follows immediately from Corollary 2.7 and Proposition 2.22. ■

Further specialization to $P_2(\varepsilon)$ is straightforward.

We call $R: E^k \to E^n$ a "polyhedral homogeneous convex" point-to-set map on a convex polyhedral cone $K \subset E^k$ if $G(R) \cap (K \times E^n)$ is a convex polyhedral cone (i.e., a convex cone and a convex polyhedron). This notion was introduced by Rockafellar (1967), who calls it a "polyhedral convex process." [He assumes that $(0,0) \in G(R) \cap (K \times E^n)$].

Proposition 2.26

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly positively homogeneous and polyhedral convex on $E^n \times K$,

R is polyhedral homogeneous convex on K , and K is a polyhedral convex cone, then f^* is positively homogeneous and polyhedral convex on K .

Proof: Follows from Propositions 2.19 and 2.21. ■

This result can also be easily specialized to the parametric NLP problems $P_3(\varepsilon)$ and $P_2(\varepsilon)$ in a straightforward manner.

3. CONCAVITY OF THE OPTIMAL VALUE FUNCTION

Recall from Section 2 that an (extended) function $\phi: E^r \to E^1 \cup \{-\infty,\infty\}$ is called "concave" on a convex set $M \subset E^r$ if $-\phi$ is convex on M , i.e., if for all $x_1,x_2 \in M$ and $\lambda \in (0,1)$,

$$\phi\big(\lambda x_1 + (1-\lambda)x_2\big) \geqslant \lambda\phi(x_1) + (1-\lambda)\phi(x_2) .$$

(here the convention is that $(-\infty) + \infty = -\infty$). We call $R: E^k \to E^n$ a "concave" point-to-set map on a convex set $S \subset E^k$ if, for all $\varepsilon_1,\varepsilon_2 \in S$ and $\lambda \in (0,1)$,

$$R\big(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) \subset \lambda R(\varepsilon_1) + (1-\lambda)R(\varepsilon_2) .$$

This definition was given by, e.g., Tagawa (1978). [A more general notion of a "B-concave" point-to-set map was introduced by Tanino and Sawaragi (1979)].

Proposition 3.1

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly concave on $E^n \times S$, R is concave on S and S is a convex set, then f^* is concave on S .

Proof: Let $\varepsilon_1,\varepsilon_2 \in S$ and $\lambda \in (0,1)$. Then by our assumptions we have

$$\begin{aligned}
f^*\big(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) &= \inf_{x\in R(\lambda\varepsilon_1+(1-\lambda)\varepsilon_2)} f\big(x,\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) \\
&\geq \inf_{\substack{x_1\in R(\varepsilon_1)\\ x_2\in R(\varepsilon_2)}} f\big(\lambda x_1 + (1-\lambda)x_2,\ \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) \\
&\geq \inf_{\substack{x_1\in R(\varepsilon_1)\\ x_2\in R(\varepsilon_2)}} [\lambda f(x_1,\varepsilon_1) + (1-\lambda)f(x_2,\varepsilon_2)] \\
&= \lambda \inf_{x_1\in R(\varepsilon_1)} f(x_1,\varepsilon_1) + (1-\lambda) \inf_{x_2\in R(\varepsilon_2)} f(x_2,\varepsilon_2) \\
&= \lambda f^*(\varepsilon_1) + (1-\lambda)f^*(\varepsilon_2) \ ,
\end{aligned}$$

(in the case when $\varepsilon_1 \in \mathrm{dom}R$ and $\varepsilon_2 \notin \mathrm{dom}R$, $\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2 \notin \mathrm{dom}R$ and the above inequality is still valid), i.e., f^* is concave on S . ■

Remark 3.2

Note that the intermediate function in the above proof given by

$$\hat{f}^*(\lambda) = \inf_{\substack{x_1\in R(\varepsilon_1)\\ x_2\in R(\varepsilon_2)}} f\big(\lambda x_1 + (1-\lambda)x_2,\ \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) \ ,$$

where $\varepsilon_1,\varepsilon_2 \in S$ are fixed, is concave on $[0,1]$ in view of Proposition 3.9 [since $R(\varepsilon_1),R(\varepsilon_2)$ are fixed and $f\big(\lambda x_1 + (1-\lambda)x_2,\ \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big)$ is concave in λ for any fixed x_1,x_2]. Therefore $\hat{f}^*$ is a concave lower bound on f^* on the interval $[\varepsilon_1,\varepsilon_2]$, hence a better bound than the linear one given by $\ell(\lambda) = \lambda f^*(\varepsilon_1) + (1-\lambda)f^*(\varepsilon_2)$.

In order to strengthen the Proposition 3.1, we introduce the following notion, paralleling one introduced in Section 2. The point-to-set map $R\colon E^k \to E^n$ is called "hull concave" on a convex set $S \subset E^k$ if the map convR given by $\text{conv}R(\varepsilon) = \text{conv}\big(R(\varepsilon)\big)$ is concave on S, or equivalently, if for all $\varepsilon_1, \varepsilon_2 \in S$ and $\lambda \in (0,1)$,

$$\text{conv}\big(R(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2)\big) \subset \lambda\text{conv}\big(R(\varepsilon_1)\big) + (1-\lambda)\text{conv}\big(R(\varepsilon_2)\big) .$$

Since the set on the right-hand side of the above inclusion is convex, we can instead use the condition

$$R\big(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) \subset \lambda\text{conv}\big(R(\varepsilon_1)\big) + (1-\lambda)\text{conv}\big(R(\varepsilon_2)\big) .$$

This shows immediately that if R is concave on a convex set S, then it is also hull concave on S. The following simple example shows that the converse statement is not true.

Example 3.3

Consider $R: E^1 \to E^1$ defined by

$$R(\varepsilon) = \begin{cases} \phi & , \text{ if } \varepsilon < 0 \\ \{0\} \cup \{1\} & , \text{ if } \varepsilon = 0 \\ \{1/2\} & , \text{ if } \varepsilon > 0 \end{cases}$$

and $S = \{\varepsilon \mid 0 \leq \varepsilon \leq 1\}$. Then it is easily seen that R is hull concave on S, but is not concave on S.

However, if R is convex-valued on S (i.e., $R(\varepsilon)$ is convex for each $\varepsilon \in S$), then the notions of concavity and hull concavity on S coincide.

Proposition 3.4

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly concave on $E^n \times S$, R is hull concave on S, and S is convex, then f^* is concave on S.

Proof: Define $\tilde{R}(\varepsilon) = \text{conv}\big(R(\varepsilon)\big)$ for $\varepsilon \in S$. Denote $\tilde{f}^*(\varepsilon) = \inf_x\{f(x,\varepsilon) \mid x \in \tilde{R}(\varepsilon)\}$. By Proposition 3.1 $\tilde{f}^*$ is concave on S, since $\tilde{R}$ is concave on S. We shall show that $\tilde{f}^*(\varepsilon) = f^*(\varepsilon)$ for $\varepsilon \in S$. Obviously, $f^*(\varepsilon) \geqslant \tilde{f}^*(\varepsilon)$ for $\varepsilon \in S$. Suppose that $\varepsilon \in S$ and $x \in \tilde{R}(\varepsilon)$. Then $x = \lambda x_1 + (1-\lambda)x_2$ for some $x_1, x_2 \in R(\varepsilon)$ and $\lambda \in (0,1)$. By the concavity of f,

$$f(x,\varepsilon) = f\big(\lambda x_1 + (1-\lambda)x_2, \varepsilon\big) \geqslant \lambda f(x_1,\varepsilon) + (1-\lambda)f(x_2,\varepsilon)$$

$$\geqslant \min\{f(x_1,\varepsilon),\ f(x_2,\varepsilon)\} \geqslant \inf_{x \in R(\varepsilon)} f(x,\varepsilon) = f^*(\varepsilon) .$$

This shows that $\tilde{f}^*(\varepsilon) = \inf_{x \in \tilde{R}(\varepsilon)} f(x,\varepsilon) \geqslant f^*(\varepsilon)$. ■

Remark 3.5

The above proof basically shows that if f is concave (or quasi-concave) in x and $R(\varepsilon)$ is an arbitrary set, then

$$\tilde{f}^*(\varepsilon) = \inf_{x \in \text{conv}(R(\varepsilon))} f(x,\varepsilon) = \inf_{x \in R(\varepsilon)} f(x,\varepsilon) = f^*(\varepsilon) .$$

This also implies that if $x^* \in S^*(\varepsilon)$, i.e., x^* is a solution to the problem $P(\varepsilon)$, then it is also a solution to the problem

$$\min_x \ f(x,\varepsilon) \quad \text{s.t.} \quad x \in \text{conv}\big(R(\varepsilon)\big) \qquad \tilde{P}(\varepsilon)$$

We also note that Remark 3.2 remains valid in this case, since for any $\varepsilon_1, \varepsilon_2 \in S$ and $\lambda \in (0,1)$, we have

$$\lambda\,\text{conv}\big(R(\varepsilon_1)\big) + (1-\lambda)\,\text{conv}\big(R(\varepsilon_2)\big) = \text{conv}\big(\lambda R(\varepsilon_1) + (1-\lambda)R(\varepsilon_2)\big) ,$$

and by Remark 3.5,

$$\inf_{x \in \text{conv}(\lambda R(\varepsilon_1)+(1-\lambda)R(\varepsilon_2))} f\big(x,\ \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big)$$

$$= \inf_{x \in \lambda R(\varepsilon_1)+(1-\lambda)R(\varepsilon_2)} f\big(x,\ \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) .$$

Similar to the convex case, we define the following notions. We call $R: E^k \to E^n$ a "weakly closure concave" point-to-set map on a convex set $S \subset E^k$ if, for all $\varepsilon_1, \varepsilon_2 \in S$ and $\lambda \in (0,1)$,

$$R(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2) \subset \lambda c\ell(R(\varepsilon_1)) + (1-\lambda)c\ell(R(\varepsilon_2)) .$$

We also call R "closure concave" on a convex set S if the map $c\ell R$ [given by $c\ell R(\varepsilon) = c\ell(R(\varepsilon))$] is concave on S. It follows that if R is concave on S, then it is weakly closure concave on S, and if $R(\varepsilon)$ is closed for all $\varepsilon \in S$, then all these three notions coincide. Also, if R is closure concave on S, then it is weakly closure concave on S, but not conversely (since in general the inclusion $c\ell(A+B) \subset c\ell(A) + c\ell(B)$ is not true even if A and B are convex sets, unless A or B is bounded).

Proposition 3.6

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly concave on $E^n \times S$ and upper semicontinuous in x on E^n for every $\varepsilon \in S$, R is weakly closure concave or closure concave on S and S is a convex set, then f^* is concave on S.

Proof: Let $\varepsilon_1, \varepsilon_2 \in S$, $\lambda \in (0,1)$, and denote $\varepsilon_\lambda = \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2$. Then by our assumptions we have

$$f^*(\varepsilon_\lambda) = \inf_{x \in R(\varepsilon_\lambda)} f(x,\varepsilon_\lambda) \geq \inf_{\substack{x_1 \in c\ell(R(\varepsilon_1)) \\ x_2 \in c\ell(R(\varepsilon_2))}} f(\lambda x_1 + (1-\lambda)x_2, \varepsilon_\lambda)$$

$$\geq \inf_{\substack{x_1 \in c\ell(R(\varepsilon_1)) \\ x_2 \in c\ell(R(\varepsilon_2))}} [\lambda f(x_1,\varepsilon_1) + (1-\lambda) f(x_2,\varepsilon_2)]$$

$$= \lambda \inf_{x_1 \in c\ell(R(\varepsilon_1))} f(x_1,\varepsilon_1) + (1-\lambda) \inf_{x_2 \in c\ell(R(\varepsilon_2))} f(x_2,\varepsilon_2) .$$

In view of Remark 2.5 we obtain further that

$$f^*(\varepsilon_\lambda) \geq \lambda \inf_{x_1 \varepsilon R(\varepsilon_1)} f(x_1,\varepsilon_1) + (1-\lambda) \inf_{x_2 \varepsilon R(\varepsilon_2)} f(x_2,\varepsilon_2)$$

$$= \lambda f^*(\varepsilon_1) + (1-\lambda) f^*(\varepsilon_2)$$

(in the case when $\varepsilon_1 \in \mathrm{dom}R$ and $\varepsilon_2 \notin \mathrm{dom}R$, the above inequality remains valid), i.e., f^* is concave on S . ■

Remark 3.7

We observe that the notions of hull concavity and closure concavity of R can be combined by considering the sets $c\ell\big(\mathrm{conv}(R(\varepsilon))\big)$. The analogue of Propositions 3.4 and 3.6 can be easily proven by combining these assumptions and noting that the closure of a convex set is also a convex set.

It is rather difficult to obtain specializations of Proposition 3.1 to the problems $P_3(\varepsilon)$ and $P_2(\varepsilon)$, as was possible in the convex case. However, the following result is an example of an application of this proposition.

For a convex function $\phi: E^r \to E^1$, the "subdifferential of ϕ at x_0 " [see, e.g., Rockafellar (1970)] is defined as

$$\partial\phi(x_0) = \{a \in E^r \mid \phi(x) - \phi(x_0) \geq a^T(x-x_0),\ \forall\ x \in E^r\} .$$

Any vector $a \in \partial\phi(x_0)$ is called a "subgradient of ϕ at x_0 ." It is well known that $\partial\phi(x) \neq \phi$ for all $x \in E^r$ if ϕ is finite and convex on E^r and that $\phi(x_0) = \min_x\{\phi(x) \mid x \in E^r\}$ if and only if $0 \in \partial\phi(x_0)$ [see, e.g., Rockafellar (1970)]. Consider the problem

$$\min_{(x,z)\varepsilon M\times E^s} f(x,z,\varepsilon) \quad \text{s.t.} \quad g_1(x,\varepsilon) + \bar{g}_1(z) \geq 0 \qquad P_3'(\varepsilon)$$

Proposition 3.8

If f is jointly concave in (z,ε) on $E^s \times S$ for every $x \in M$, g_1 is convex in ε on S for every $x \in M$, $\bar{g}_1$ is convex on E^r and such that for all $z \in E^s$, $0 \notin \partial\bar{g}_1(z)$, S is convex and M is arbitrary, then f^*, the optimal value function of $P_3'(\varepsilon)$, is concave on S.

Proof: We shall prove that the feasible set map $R'(\varepsilon) = \{(x,z,v) \mid f(x,z,\varepsilon) \leq v,\ g_1(x,\varepsilon) + \bar{g}_1(z) \geq 0,\ x \in M\}$ of the equivalent problem

$$\min_{(x,z,v)} \quad v \quad \text{s.t.} \quad (x,z,v) \in R'(\varepsilon) \qquad \bar{P}_3'(\varepsilon)$$

is concave on S, and thus by Proposition 3.1 v^*, the optimal value function of $\bar{P}_3'(\varepsilon)$, will be concave on S. This will imply concavity of f^* on S, since $f^* = v^*$.

Let $(\bar{x},\bar{z},\bar{v}) \in R'(\varepsilon_\lambda)$, where $\varepsilon_1,\varepsilon_2 \in S$, $\varepsilon_\lambda = \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2$, $\lambda \in (0,1)$. Then $\bar{x} \in M$, $f(\bar{x},\bar{z},\varepsilon_\lambda) \leq \bar{v}$, $g_1(\bar{x},\varepsilon_\lambda) + \bar{g}_1(\bar{z}) \geq 0$. By the convexity and finiteness of $\bar{g}_1$ on E^r, $\partial\bar{g}_1(\bar{z}) \neq \phi$. Let $a \in \partial\bar{g}_1(\bar{z})$. Then for all $z \in E^r$, $\bar{g}_1(z) - \bar{g}_1(\bar{z}) \geq a^T(z-\bar{z})$. Denote $b = \bar{g}_1(\bar{z}) - a^T\bar{z}$. Then we have, for all $z \in E^r$, $\bar{g}_1(z) \geq a^Tz + b$ and also $g_1(\bar{x},\varepsilon_\lambda) + a^T\bar{z} + b \geq 0$. By convexity of g_1 in ε,

$$\Delta_\lambda = \lambda g_1(\bar{x},\varepsilon_1) + (1-\lambda)g_1(\bar{x},\varepsilon_2) + a^T\bar{z} + b \geq 0 .$$

Let $z_0 \in E^s$ be such that $a^Tz_0 = 0$ and $\bar{z} = z_0 + (a^T\bar{z}/a^Ta)a$. Such a vector z_0 exists, since $a \neq 0$, by our assumptions. Let $z_i = z_0 + (\Delta_\lambda - g_1(\bar{x},\varepsilon_i) - b)(a/(a^Ta))$, for $i = 1,2$. Then it follows that $\bar{z} = \lambda z_1 + (1-\lambda)z_2$ and for $i = 1,2$, $g_1(\bar{x},\varepsilon_i) + \bar{g}_1(z_i) \geq g_1(\bar{x},\varepsilon_i) + a^Tz_i + b = \Delta_\lambda \geq 0$. By concavity of f in (z,ε),

$$c_\lambda = \bar{v} - \lambda f(\bar{x}, z_1, \varepsilon_1) - (1-\lambda) f(\bar{x}, z_2, \varepsilon_2) \geqslant 0 .$$

Let $v_i = f(\bar{x}, z_i, \varepsilon_i) + c_\lambda$ for $i = 1,2$. Then $\bar{v} = \lambda v_1 + (1-\lambda) v_2$ and $v_i \geqslant f(\bar{x}, z_i, \varepsilon_i)$, $i = 1,2$. Thus, $(\bar{x}, \bar{z}, \bar{v}) = \lambda(\bar{x}, z_1, v_1) + (1-\lambda)(\bar{x}, z_2, v_2)$ and, in view of the above relationships, $(\bar{x}, z_1, v_1) \in R'(\varepsilon_1)$ and $(\bar{x}, z_2, v_2) \in R'(\varepsilon_2)$, completing the proof of the concavity of R' on S. ■

Note that the condition $0 \notin \partial \bar{g}_1(z)$ for all $z \in E^r$ holds if and only if $\bar{g}_1$ does not attain a minimum on E^r.

The next proposition is apparently well known.

<u>Proposition 3.9</u>

Consider the special problem $P_1(\varepsilon)$. If f is concave in ε on S for all $x \in M$, S is convex and M is arbitrary, then f^* is concave on S.

<u>Proof</u>: Let $\varepsilon_1, \varepsilon_2 \in S$, $\lambda \in (0,1)$. By our assumptions,

$$\begin{aligned} f^*\big(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) &= \inf_{x \in M} f\big(x, \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) \\ &\geqslant \inf_{x \in M} [\lambda f(x,\varepsilon_1) + (1-\lambda) f(x,\varepsilon_2)] \\ &\geqslant \lambda \inf_{x \in M} f(x,\varepsilon_1) + (1-\lambda) \inf_{x \in M} f(x,\varepsilon_2) \\ &= \lambda f^*(\varepsilon_1) + (1-\lambda) f^*(\varepsilon_2) , \end{aligned}$$

i.e., f^* is concave on S. ■

The following rather special result partially extends Proposition 3.9.

Proposition 3.10

Consider the general parametric optimization problem $P(\varepsilon)$. If f is quasiconcave in x on M for all $\varepsilon \in S$ and is concave in ε on S for all $x \in M$, S is convex and $\text{conv}(R(\varepsilon)) = M$ for all $\varepsilon \in S$, then f^* is concave on S.

Proof: Define $\tilde{f}^*(\varepsilon) = \inf_x\{f(x,\varepsilon) \mid x \in M\}$ for $\varepsilon \in S$. By Proposition 3.9, $\tilde{f}^*$ is concave on S. In view of Remark 3.5, $\tilde{f}^*(\varepsilon) = \inf_{x\in R(\varepsilon)} f(x,\varepsilon) = f^*(\varepsilon)$ for $\varepsilon \in S$; thus f^* is also concave on S.

■

Realizations of Proposition 3.9 are frequently used, in particular when f is linear in ε. For example, in linear programming, if $f(x,\varepsilon) = \varepsilon^T x$ and $M = \{x \geq 0 \mid Ax \geq b\}$ for some matrix A and vector b, then $f^*(\varepsilon) = \inf_x\{f(x,\varepsilon) \mid x \in M\}$ is concave; see also Proposition 3.11. Note that Proposition 3.9 is a version of a more general statement, that $\inf_{i\in I} f_i$ is concave if $\{f_i\}$ are concave for $i \in I$, where I is an arbitrary index set (and also can be viewed as a corollary of Proposition 3.8).

As another application of this result, consider the Wolfe dual of $P_2(\varepsilon)$ [Wolfe (1961)], with $f(x,\varepsilon) = f(x)$ (i.e., the standard rhs parametric NLP problem),

$$\max_{(x,u,w)} \; L(x,u,w,\varepsilon) \quad \text{s.t.} \quad \nabla_x L(x,u,w,\varepsilon) = 0, \quad u \geq 0,\ x \in M \qquad D_2(\varepsilon)$$

where L is the Lagrangian function given by

$$L(x,u,w,\varepsilon) = f(x) - \sum_{i=1}^{m} u_i[g_i(x) - \varepsilon_i] + \sum_{j=1}^{p} w_j[h_j(x) - \varepsilon_{m+j}] .$$

Denote by $L^*(\varepsilon)$ the optimal value of $D_2(\varepsilon)$. Note that although the Wolfe dual is usually defined under convexity assumptions on $P_2(\varepsilon)$, convexity assumptions are not needed in the following result.

<u>Proposition 3.11</u>

For arbitrary functions f, $\{g_i\}$, and $\{h_j\}$ and an arbitrary set M, L^* is convex.

<u>Proof</u>: The function

$$L^*(\varepsilon) = \sup_{(x,u,w)\varepsilon R_D(\varepsilon)} L(x,u,w,\varepsilon) = -\inf_{(x,u,w)\varepsilon R_D(\varepsilon)} [-L(x,u,w,\varepsilon)] ,$$

where $R_D(\varepsilon) = \{(x,u,w) \mid \nabla_x L(x,u,w,\varepsilon) = 0,\ u \geqslant 0,\ x \in M\}$. Since $-L$ is linear in ε and R_D is constant, the result follows from Proposition 3.9. ∎

<u>Remark 3.12</u>

This result can be further extended to the dual of the problem

$$\min_x f(x) \quad \text{s.t.} \quad \begin{array}{ll} g_i(x) \geqslant \bar{g}_i(\varepsilon) , & i = 1,\dots,m \\ h_j(x) = \bar{h}_j(\varepsilon) , & j = 1,\dots,p \end{array} \qquad P_2'(\varepsilon)$$

under the assumptions that the functions $\{\bar{g}_i\}$ are convex and that $\{\bar{h}_j\}$ are affine.

Recalling the definition in Section 2, we call a function $\phi: E^r \to E^1 \cup \{-\infty,\infty\}$ "concave at x_0" on a set $M \subset E^r$ convex at x_0, if for all $x \in M$ and $\lambda \in (0,1)$,

$$\phi\big(\lambda x_0 + (1-\lambda)x\big) \geqslant \lambda\phi(x_0) + (1-\lambda)\phi(x) .$$

We also call the point-to-set map R "concave at ε_0" on a set S convex at ε_0 if, for all $\varepsilon \in S$ and $\lambda \in (0,1)$,

$$R\left(\lambda\varepsilon_0 + (1-\lambda)\varepsilon\right) \subset \lambda R(\varepsilon_0) + (1-\lambda)R(\varepsilon) .$$

Furthermore, we call R "hull concave at ε_0 " on a set S convex at ε_0 if the map convR is concave at ε_0 on S .

The next two propositions parallel Propositions 3.1 (or 3.4) and 3.9.

Proposition 3.13

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly concave at (x,ε_0) for all $x \in R(\varepsilon_0)$ on $E^n \times S$, R is hull concave or concave at ε_0 on S , and S is convex at ε_0 , then f^* is concave at ε_0 on S .

Proof: Analogous to the proof of Proposition 3.1 (resp. Proposition 3.4) if R is concave at ε_0 (hull concave at ε_0). ■

Proposition 3.14

Consider the special problem $P_1(\varepsilon)$. If f is concave in ε at ε_0 on S for any $x \in M$, S is convex at ε_0 , and M is arbitrary, then f^* is concave at ε_0 on S .

Proof: Similar to the proof of Proposition 3.9. ■

We now consider the notion of strict concavity. A function $\phi: E^r \to E^1 \cup \{-\infty,\infty\}$ is called "strictly concave" on a convex set $M \subset E^r$ if $-\phi$ is strictly convex on M , or equivalently, if for all $x_1,x_2 \in M$, $x_1 \neq x_2$ and $\lambda \in (0,1)$,

$$\phi\left(\lambda x_1 + (1-\lambda)x_2\right) > \lambda\phi(x_1) + (1-\lambda)\phi(x_2) .$$

Proposition 3.15

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly strictly concave on $E^n \times S$, R is hull concave or concave on S, S is convex, and $S^*(\varepsilon) \neq \phi$ for all $\varepsilon \in S$, then f^* is strictly concave on S.

Proof: In view of Remark 3.5 it is enough to prove the result for R concave. Let $\varepsilon_1, \varepsilon_2 \in S$, $\varepsilon_1 \neq \varepsilon_2$, and $\lambda \in (0,1)$. Denote $\varepsilon_\lambda = \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2$. By our assumptions there exists $x^* \in S^*(\varepsilon_\lambda)$ and by the concavity of R, $x^* = \lambda x_1 + (1-\lambda)x_2$ for some $x_1 \in R(\varepsilon_1)$, $x_2 \in R(\varepsilon_2)$. Using strict concavity of f we obtain

$$
\begin{aligned}
f^*(\varepsilon_\lambda) = f(x^*,\varepsilon_\lambda) &= f\big(\lambda x_1 + (1-\lambda)x_2, \varepsilon_\lambda\big) \\
&> \lambda f(x_1,\varepsilon_1) + (1-\lambda) f(x_2,\varepsilon_2) \\
&\geqslant \lambda \inf_{x \in R(\varepsilon_1)} f(x,\varepsilon_1) + (1-\lambda) \inf_{x \in R(\varepsilon_2)} f(x,\varepsilon_2) \\
&= \lambda f^*(\varepsilon_1) + (1-\lambda) f^*(\varepsilon_2) ,
\end{aligned}
$$

i.e., f^* is strictly concave on S. ■

In order to obtain the next result we introduce the following notion. We call a point-to-set map $R: E^k \to E^n$ "strictly concave" on a convex set $S \subset E^k$ if for any $\varepsilon_1, \varepsilon_2 \in S$, $\varepsilon_1 \neq \varepsilon_2$, and $\lambda \in (0,1)$ and $x \in R\big(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big)$ there exist $x_1 \in R(\varepsilon_1)$ and $x_2 \in R(\varepsilon_2)$ such that $x_1 \neq x_2$ and $x = \lambda x_1 + (1-\lambda)x_2$. It is clear that if R is strictly concave on S, then it is concave on S. However, if R is concave on S it need not be strictly concave on S. For example, if $R(\varepsilon) = M$ for $\varepsilon \in S$, where M is arbitrary, then R is concave on S, but it is strictly concave on S only if M does not have any extreme points [$x_0 \varepsilon M$ is called an "extreme point of M" if $x_0 = \lambda x_1 + (1-\lambda)x_2$ for some $x_1, x_2 \varepsilon M$ and $\lambda \varepsilon (0,1)$ implies that $x_1 = x_2$]. Also, we call R "strictly hull concave" on a convex set S if the map convR is strictly concave on S.

Proposition 3.16

Consider the parametric optimization problem $P'(\varepsilon)$. If f is strictly concave on E^n, R is strictly hull concave or strictly concave on S, S is convex, and $S^*(\varepsilon) \neq \phi$ for all $\varepsilon \varepsilon S$, then f^* is strictly concave on S.

Proof: By Remark 3.5 if is enough to give the proof for the case when R is strictly concave. Let $\varepsilon_1, \varepsilon_2 \varepsilon S$, $\varepsilon_1 \neq \varepsilon_2$, $\lambda \varepsilon (0,1)$, and $\varepsilon_\lambda = \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2$. By our assumptions, there exists $x^* \varepsilon S^*(\varepsilon_\lambda)$ and, by strict concavity of R, $x^* = \lambda x_1 + (1-\lambda)x_2$ for some $x_1 \varepsilon R(\varepsilon_1)$, $x_2 \varepsilon R(\varepsilon_2)$, $x_1 \neq x_2$. Using strict concavity of f we have

$$f^*(\varepsilon_\lambda) = f(x^*) = f\big(\lambda x_1 + (1-\lambda)x_2\big) > \lambda f(x_1) + (1-\lambda)f(x_2)$$

$$\geq \lambda \inf_{x \varepsilon R(\varepsilon_1)} f(x) + (1-\lambda) \inf_{x \varepsilon R(\varepsilon_2)} f(x) = \lambda f^*(\varepsilon_1) + (1-\lambda)f^*(\varepsilon_2),$$

i.e., f^* is strictly concave on S. ■

For the special problem $P_1(\varepsilon)$ we obtain the following result.

Proposition 3.17

Consider the problem $P_1(\varepsilon)$. If f is strictly concave in ε on S for all $x \varepsilon M$, S is convex, M is arbitrary, and $S^*(\varepsilon) \neq \phi$ for all $\varepsilon \varepsilon S$, then f^* is strictly concave on S.

Proof: Let $\varepsilon_1, \varepsilon_2 \varepsilon S$, $\varepsilon_1 \neq \varepsilon_2$, $\lambda \varepsilon (0,1)$, and $\varepsilon_\lambda = \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2$. By the assumptions there exists $x^* \varepsilon S^*(\varepsilon_\lambda)$. Using strict concavity of f we obtain

$$f^*(\varepsilon_\lambda) = f(x^*,\varepsilon_\lambda) > \lambda f(x^*,\varepsilon_1) + (1-\lambda)f(x^*,\varepsilon_2)$$

$$\geq \lambda \inf_{x\in M} f(x,\varepsilon_1) + (1-\lambda) \inf_{x\in M} f(x,\varepsilon_2) = \lambda f^*(\varepsilon_1) + (1-\lambda)f^*(\varepsilon_2) ,$$

i.e., f^* is strictly concave on S . ∎

As a counterpart to the notion of uniform convexity considered in Section 2, we consider the notion of uniform concavity. We call $\phi: E^r \to E^1 \cup \{-\infty,\infty\}$ "uniformly concave with $d(\cdot)$ " on a convex set $M \subset E^r$ if $-\phi$ is uniformly convex with $d(\cdot)$ on M , i.e., if for all $x_1,x_2 \in M$ and $\lambda \in (0,1)$,

$$\phi\big(\lambda x_1 + (1-\lambda)x_2\big) \geq \lambda\phi(x_1) + (1-\lambda)\phi(x_2) + \lambda(1-\lambda)d(\|x_1-x_2\|) ,$$

where $d: [0,\infty) \to [0,\infty)$ is an increasing function with $d(t) > 0$ for $t > 0$ and $d(0) = 0$, and $\|\cdot\|$ is an arbitrary norm in E^r .

Proposition 3.18

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly uniformly concave with $d(\cdot)$ on $E^n \times S$, R is hull concave or concave on S , and S is a convex set, then f^* is uniformly concave with $d(\cdot)$ on S .

Proof: In view of Remark 3.5, we prove the result only for R concave. Let $\varepsilon_1,\varepsilon_2 \in S$ and $\lambda \in (0,1)$. By our assumptions and the properties of the function d and the norm, we have

$$f^*\big(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) = \inf_{x\in R(\lambda\varepsilon_1+(1-\lambda)\varepsilon_2)} f\big(x, \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big)$$

$$\geq \inf_{\substack{x_1\in R(\varepsilon_1)\\ x_2\in R(\varepsilon_2)}} f\big(\lambda x_1 + (1-\lambda)x_2, \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big)$$

$$\geq \inf_{\substack{x_1 \varepsilon R(\varepsilon_1) \\ x_2 \varepsilon R(\varepsilon_2)}} [\lambda f(x_1,\varepsilon_1) + (1-\lambda)f(x_2,\varepsilon_2) + \lambda(1-\lambda)d(\|(x_1,\varepsilon_1)-(x_2,\varepsilon_2)\|)]$$

$$= \inf_{\substack{x_1 \varepsilon R(\varepsilon_1) \\ x_2 \varepsilon R(\varepsilon_2)}} [\lambda f(x_1,\varepsilon_1) + (1-\lambda)f(x_2,\varepsilon_2) + \lambda(1-\lambda)d(\|(x_1-x_2,\ \varepsilon_1-\varepsilon_2)\|)]$$

$$\geq \inf_{\substack{x_1 \varepsilon R(\varepsilon_1) \\ x_2 \varepsilon R(\varepsilon_2)}} [\lambda f(x_1,\varepsilon_1) + (1-\lambda)f(x_2,\varepsilon_2)] + \lambda(1-\lambda)d(\|\varepsilon_1-\varepsilon_2\|)$$

$$= \lambda \inf_{x_1 \varepsilon R(\varepsilon_1)} f(x,\varepsilon_1) + (1-\lambda) \inf_{x_2 \varepsilon R(\varepsilon_2)} f(x_2,\varepsilon_2) + \lambda(1-\lambda)d(\|\varepsilon_1-\varepsilon_2\|)$$

$$= \lambda f^*(\varepsilon_1) + (1-\lambda)f^*(\varepsilon_2) + \lambda(1-\lambda)d(\|\varepsilon_1-\varepsilon_2\|)$$

(if $\varepsilon_1 \in \mathrm{dom}R$ and $\varepsilon_2 \notin \mathrm{dom}R$ the inequality above remains valid), i.e., f^* is uniformly concave with $d(\cdot)$ on S. ■

Note that for the parametric optimization problem $P'(\varepsilon)$, this result reduces to the previous one in Proposition 3.1, namely, the concavity of f^*. Also, see the remarks following Proposition 2.17 concerning uniform convexity, since they are applicable in this case as well.

<u>Proposition 3.19</u>

Consider the special problem $P_1(\varepsilon)$. If f is uniformly concave in ε with $d(\cdot)$ on S for all $x \in M$, S is convex and M is arbitrary, then f^* is uniformly concave with $d(\cdot)$ on S.

<u>Proof</u>: Let $\varepsilon_1,\varepsilon_2 \in S$, $\lambda \in (0,1)$. By our assumptions,

$$f^*(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2) = \inf_{x\in M} f(x, \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2)$$

$$\geq \inf_{x\in M} [\lambda f(x,\varepsilon_1) + (1-\lambda)f(x,\varepsilon_2) + \lambda(1-\lambda)d(\|\varepsilon_1-\varepsilon_2\|)]$$

$$\geq \lambda \inf_{x\in M} f(x,\varepsilon_1) + (1-\lambda) \inf_{x\in M} f(x,\varepsilon_2) + \lambda(1-\lambda)d(\|\varepsilon_1-\varepsilon_2\|)$$

$$= \lambda f^*(\varepsilon_1) + (1-\lambda)f^*(\varepsilon_2) + \lambda(1-\lambda)d(\|\varepsilon_1-\varepsilon_2\|) ,$$

i.e., f^* is uniformly concave with $d(\cdot)$ on S . ■

Finally, we consider the notion of homogeneous concavity introduced in Section 2. Recall that $\phi: E^r \to E^1 \cup \{-\infty,\infty\}$ is called "homogeneous concave" on a convex cone $K \subset E^r$ if it is both positively homogeneous on K , i.e., $\phi(\lambda x) = \lambda\phi(x)$ for all $x \in K$ and $\lambda > 0$, and concave on K . Analogously, a point-to-set map $R: E^k \to E^n$ will be called "homogeneous concave" on a convex cone $K \subset E^k$ if it is both concave and positively homogeneous on K . (See Section 2 for the latter notion.) Recently Ioffe (1979) introduced the notion of a "fan," which in our terminology is a convex-valued homogeneous concave point-to-set map (in a finite-dimensional space he slightly modified this definition). We additionally define R to be a "homogeneous hull concave" point-to-set map on a convex cone $K \subset E^k$ if it is both hull concave and positively homogeneous on K .

<u>Proposition 3.20</u>

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly homogeneous concave on $E^n \times K$, R is homogeneous hull concave or homogeneous concave on K , and K is a convex cone, then f^* is homogeneous concave on K .

Proof: Follows directly from Propositions 3.1 (or 3.4) and 2.21 taken together. ■

Proposition 3.21

Consider the special problem $P_1(\varepsilon)$. If f is homogeneous concave in ε on K for all $x \varepsilon M$, K is a convex cone and M is arbitrary, then f^* is homogeneous concave on K.

Proof: Follows immediately from Propositions 3.9 and 2.23. ■

4. AFFINENESS OF THE OPTIMAL VALUE FUNCTION AND CONVEXITY PROPERTIES OF THE SOLUTION SET MAP

We shall consider here the solution set map $S^*: E^k \to E^n$ for the general parametric problem $P(\varepsilon)$ given by

$$S^*(\varepsilon) = \{x \varepsilon R(\varepsilon) \mid f(x,\varepsilon) \leqslant f^*(\varepsilon)\} .$$

The following result is well known.

Proposition 4.1

Consider the general parametric optimization problem $P(\varepsilon)$. If f is quasiconvex in x for all $\varepsilon \varepsilon S$, R is convex-valued on S (i.e., $R(\varepsilon)$ is convex for any $\varepsilon \varepsilon S$), and S is an arbitrary set, then S^* is convex-valued on S.

Proof: Let $\varepsilon \varepsilon S$. Then $R(\varepsilon)$ is convex and the set $\{x \mid f(x,\varepsilon) \leqslant f^*(\varepsilon)\}$ is convex by quasiconvexity of f in x, so $S^*(\varepsilon)$ is also convex. ■

A function $\phi: E^r \to E^1$ is called "affine" on a convex set $M \subset E^r$ if, for all $x_1, x_2 \varepsilon M$ and $\lambda \varepsilon (0,1)$,

$$\phi(\lambda x_1 + (1-\lambda)x_2) = \lambda\phi(x_1) + (1-\lambda)\phi(x_2) .$$

Thus ϕ is affine on M if it is finite and both convex and concave on M. We call $R: E^k \to E^n$ an "affine" point-to-set map on a convex set $S \subset E^k$ if R is both convex and concave on S, i.e., if for all $\varepsilon_1, \varepsilon_2 \in S$ and $\lambda \in (0,1)$,

$$\lambda R(\varepsilon_1) + (1-\lambda)R(\varepsilon_2) = R\big(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) .$$

This notion was introduced and utilized by Tagawa (1978).

Note also that R is affine on S if and only if R is both convex and hull concave on S. We can slightly extend this definition and call $R: E^k \to E^n$ "essentially affine" on a convex set $S \subset E^k$ if it is both essentially convex and concave on S, i.e., if for all $\varepsilon_1, \varepsilon_2 \in S$, $\varepsilon_1 \neq \varepsilon_2$, and $\lambda \in (0,1)$,

$$\lambda R(\varepsilon_1) + (1-\lambda)R(\varepsilon_2) = R\big(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) .$$

Recently Penot (1982) also introduced this notion, calling such R an "affine" point-to-set map. It is clear that if R is affine on a convex set S, then it is essentially affine on S. The converse statement is not true, similar to the case of essential convexity of R. We also introduce the following notions. The point-to-set map $R: E^k \to E^n$ is called "hull affine" ("essentially hull affine") on a convex set $S \subset E^k$ if the map convR given by $\text{convR}(\varepsilon) = \text{conv}\big(R(\varepsilon)\big)$ is affine (essentially affine) on S, or equivalently if, for all $\varepsilon_1, \varepsilon_2 \in S$ $(\varepsilon_1 \neq \varepsilon_2)$ and $\lambda \in (0,1)$,

$$\lambda \text{conv}\big(R(\varepsilon_1)\big) + (1-\lambda)\text{conv}\big(R(\varepsilon_2)\big) = \text{conv}\big(R(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2)\big) .$$

An affine map R is also hull affine, i.e., both hull convex and hull concave. An example of a map that is hull affine but not affine is provided by the "constant" map R given by $R(\varepsilon) = M$, $\varepsilon \in S$, where S is convex and M is arbitrary, and not convex.

The following result shows that affineness of the map S^* is generally possible only under strong assumptions on f and R, assumptions which also imply that f^* is affine, if it is finite.

Proposition 4.2

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly affine on $E^n \times S$, R is (essentially) affine on S, and $S \subset \text{dom}R$ is convex, then f^* is both convex and concave on S and S^* is an (essentially) affine point-to-set map on S. If R is only (essentially) hull affine on S, then f^* is both convex and concave on S.

Proof: Convexity and concavity of f^* on S follows directly from Propositions 2.2 and 3.1 in the first case and from Propositions 2.10 and 3.4 in the second case. Now, for $\varepsilon_1, \varepsilon_2 \in S$, $\lambda \in (0,1)$, denote $\varepsilon_\lambda = \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2$ and let $x_1^* \in S^*(\varepsilon_1)$, $x_2^* \in S^*(\varepsilon_2)$, i.e., $x_1^* \in R(\varepsilon_1)$, $x_2^* \in R(\varepsilon_2)$ with $f(x_1^*,\varepsilon_1) \leqslant f^*(\varepsilon_1)$, $f(x_2^*,\varepsilon_2) \leqslant f^*(\varepsilon_2)$. By convexity of R, $\lambda x_1^* + (1-\lambda)x_2^* \in R(\varepsilon_\lambda)$. Also, by convexity of f and concavity of f^*,

$$\begin{aligned} f\left(\lambda x_1^* + (1-\lambda)x_2^*, \varepsilon_\lambda\right) &\leqslant \lambda f(x_1^*,\varepsilon_1) + (1-\lambda)f(x_2^*,\varepsilon_2) \\ &\leqslant \lambda f^*(\varepsilon_1) + (1-\lambda)f^*(\varepsilon_2) \leqslant f^*(\varepsilon_\lambda) , \end{aligned}$$

i.e., S^* is convex on S. Assuming that $\varepsilon_1 \neq \varepsilon_2$ we would only obtain that S^* is essentially convex on S. To prove concavity of S^* on S, let $x^* \in S^*(\varepsilon_\lambda)$. Then $x^* \in R(\varepsilon_\lambda)$ and, by concavity of R, $x^* = \lambda x_1 + (1-\lambda)x_2$ for some $x_1 \in R(\varepsilon_1)$, $x_2 \in R(\varepsilon_2)$. If either $x_1 \notin S^*(\varepsilon_1)$ or $x_2 \notin S^*(\varepsilon_2)$, then by concavity of f,

$$f^*(\varepsilon_\lambda) \geq f(x^*,\varepsilon_\lambda) = f\big(\lambda x_1 + (1-\lambda)x_2,\ \varepsilon_\lambda\big)$$

$$\geq \lambda f(x_1,\varepsilon_1) + (1-\lambda)f(x_2,\varepsilon_2) > \lambda f^*(\varepsilon_1) + (1-\lambda)f^*(\varepsilon_2) ,$$

contradicting convexity of f^* . Thus, $x_1 \varepsilon S^*(\varepsilon_1)$ and $x_2 \varepsilon S^*(\varepsilon_2)$, with $x^* = \lambda x_1 + (1-\lambda)x_2$, completing the proof. ∎

Remark 4.3

Note that convexity of S^* follows under assumptions that f and R are convex and f^* is concave, but this is possible essentially only if both f and R are affine.

Corollary 4.4

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly affine on $E^n \times S$, R is polyhedral convex and affine on S , and $S \subset \text{dom}R$ is a convex polyhedron, then f^* is both polyhedral convex and concave on S and S^* is polyhedral convex and affine on S .

Proof: In view of Propositions 2.19 and 4.2, we need only prove that S^* is polyhedral convex on S , i.e., that $G(S^*) \cap (S \times E^n)$ is a convex polyhedron. Note that

$$G(S^*) \cap (S \times E^n) = G(R) \cap (S \times E^n) \cap \{(\varepsilon,x) \mid f(x,\varepsilon) \leq f^*(\varepsilon)\} .$$

Since R is polyhedral convex on S , f is affine on S and f^* is both convex and concave on S , both sets $G(R) \cap (S \times E^n)$ and $\{(\varepsilon,x) \mid f(x,\varepsilon) \leq f^*(\varepsilon)\}$ are convex polyhedra, hence so is the set $G(S^*) \cap (S \times E^n)$, thus proving the result. ∎

Note that if f^* is finite in Corollary 4.4, then it is affine. Note also that if R is affine on a convex polyhedron S , then in general R is not polyhedral convex on S . For instance, if $R(\varepsilon) = M$,

$\varepsilon \in S$, S is a convex polyhedron, and M is a convex set but not a convex polyhedron, then R is affine on S but $G(R) \cap (S \times E^n)$ is not a convex polyhedron.

The following generalization of the notion of a polyhedral convex point-to-set map R was introduced by Robinson (1981). A point-to-set map $R: E^k \to E^n$ is called "polyhedral" if $G(R)$ is a union of finitely many convex polyhedra. Consider a convex quadratic programming problem

$$\min_{x \in E^n} \frac{1}{2} x^T Cx + \varepsilon_1^T x \quad \text{s.t.} \quad Ax \geq \varepsilon_2, \; x \geq 0 \; , \qquad QP(\varepsilon)$$

where C is a positive semidefinite $n \times n$ matrix, A is an arbitrary $m \times n$ matrix, and $\varepsilon = (\varepsilon_1, \varepsilon_2)^T \in E^{n+m}$ is the parameter vector. Robinson (1981) proves that the solution set map S^* for $QP(\varepsilon)$ is polyhedral.

Combining the notions of closure convexity and closure concavity we call $R: E^k \to E^n$ a "closure affine" point-to-set map on a convex set $S \subset E^k$ if the map $c\ell R$ is affine on S .

Proposition 4.5

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly affine on $E^n \times S$ and upper semicontinuous in x on the set $c\ell(R(S))$ for every $\varepsilon \in S$, R is closure affine on S and $S \subset \text{dom}R$ is convex, then f^* is both convex and concave on S .

Proof: Follows directly from Proposition 4.2 and Remark 2.5 taken together. ■

Affineness of S^* does not seem to follow from the above assumptions, since in general the inclusion $S^*(\varepsilon) \subset \bar{S}^*(\varepsilon) = \{x \in c\ell(R(\varepsilon)) \mid f(x,\varepsilon) \leq \bar{f}^*(\varepsilon)\}$, $\varepsilon \in S$, is strict (this inclusion is a consequence of Remark 2.5).

Recall from Section 2 that we call $\phi: E^r \to E^1$ "affine at x_0" on a set $M \subset E^r$ convex at x_0 if for all $x \in M$ and $\lambda \in (0,1)$

$$\phi\big(\lambda x_0 + (1-\lambda)x\big) = \lambda\phi(x_0) + (1-\lambda)\phi(x) .$$

Generalizing the notion of affineness for point-to-set maps, we call R "affine at ε_0," on a set S convex at ε_0, if R is both convex at ε_0 and concave at ε_0 on S, i.e., if for all $\varepsilon \in S$ and $\lambda \in (0,1)$,

$$\lambda R(\varepsilon_0) + (1-\lambda)R(\varepsilon) = R\big(\lambda\varepsilon_0 + (1-\lambda)\varepsilon\big) .$$

We shall call $R: E^k \to E^n$ "essentially affine at ε_0" on a set S convex at ε_0, if the above equality holds for all $\varepsilon \in S$, $\varepsilon \neq \varepsilon_0$ and $\lambda \in (0,1)$. This notion was also recently introduced and utilized by Penot (1982), who calls such R a "semi-affine" point-to-set map. Clearly, if R is affine at ε_0 on S, then it is essentially affine at ε_0 on S, but not vice versa (the set $R(\varepsilon_0)$ need not be convex if R is essentially affine at ε_0).

Proposition 4.6

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly affine at (x,ε_0) for all $x \in R(\varepsilon_0)$ on $E^n \times S$, R is (essentially) affine at ε_0 on S, and $S \subset \text{dom}R$ is convex at ε_0, then f^* is both convex and concave at ε_0 on S and S^* is (essentially) affine at ε_0 on S.

Proof: Convexity and concavity of f^* on S follows directly from Propositions 2.11 and 3.13, respectively. The proof of (essential) affineness of S^* at ε_0 parallels the proof of Proposition 4.2. ■

We call $R: E^k \to E^n$ a "homogeneous affine" point-to-set map on a convex cone $K \subset E^k$ if it is both affine and positively homogeneous on K [Rockafellar (1967) calls such a map "quasi-linear"]. Note that a homogeneous affine map is both homogeneous convex and affine. The following examples show that these latter notions are distinct. Let $R_1: E^1 \to E^1$ be defined by $R_1(\varepsilon) = \{x \mid 0 \leqslant x \leqslant 1\}$ for all $\varepsilon \in E^1$. Then R_1 is affine on E^1 but not homogeneous convex on E^1. Let $R_2: E^1 \to E^1$ be given by $R_2(\varepsilon) = \{x \mid x \geqslant |\varepsilon|\}$ for all $\varepsilon \in E^1$. Then R_2 is homogeneous convex on E^1 but not affine on E^1. We now give a result involving these notions.

Proposition 4.7

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly linear on $E^n \times K$, R is homogeneous affine on K, and $S \subset \mathrm{dom} R$ is a convex cone, then f^* is both homogeneous convex and homogeneous concave on K and S^* is a homogeneous affine point-to-set map on K.

Proof: Homogeneous convexity and concavity of f^* follows from Propositions 2.21 and 4.2. Affineness of S^* follows from Proposition 4.2. To prove that S^* is positively homogeneous, let $\varepsilon \in K$, $x \in S^*(\varepsilon)$, $\lambda > 0$. Since $\lambda x \in R^*(\lambda\varepsilon)$ and $f(\lambda x, \lambda\varepsilon) = \lambda f(x,\varepsilon) \leqslant \lambda f^*(\varepsilon) = f^*(\lambda\varepsilon)$, it follows that $\lambda x \in S^*(\lambda\varepsilon)$. ■

Note that in the above result f^* will be linear on K if it is finite on K, since a homogeneous and affine function is linear.

The following notion was introduced by Berge (1963) [see also Crouzeix (1973)]. A point-to-set map $R: E^k \to E^n$ is called "linear" on a linear subspace $V \subset E^k$ if $G(R) \cap (V \times E^n)$ is a linear subspace, or equivalently if, for all $\varepsilon_1, \varepsilon_2 \in V$ and $\lambda, \mu \neq 0$,

$$\lambda R(\varepsilon_1) + \mu R(\varepsilon_2) = R(\lambda\varepsilon_1 + \mu\varepsilon_2) .$$

It is immediately seen that if R is linear on a linear subspace V , then it is homogeneous affine on V . The following example shows that the converse is not true.

Example 4.8

Consider $R: E^1 \to E^1$ given by $R(\varepsilon) = \{x \mid x \leq \varepsilon\}$. Then it is easy to check that R is affine on E^1 . But $G(R)$ is a half-space in E^2 and not a linear subspace, so R is not linear on E^1 .

Proposition 4.9

Consider the general parametric optimization problem $P(\varepsilon)$. If f is jointly linear on $E^n \times V$, R is linear on V , and $V \subset \mathrm{dom}R$ is a linear subspace, then f^* if both homogeneous convex and concave on V . If, in addition, f^* is finite on V , then f^* is linear on V and S^* is a linear point-to-set map on V .

Proof: The first part of the result and homogeneous affineness of S^* follow directly from Proposition 4.7. Concerning the second part, we thus need only prove that $S^*(-\varepsilon) \subset -S^*(\varepsilon)$. Let $x^* \in S^*(-\varepsilon)$, $\varepsilon \in V$, i.e., $f(x^*,-\varepsilon) = f^*(-\varepsilon)$. Then by linearity $f(-x^*,\varepsilon) = f^*(\varepsilon)$, i.e., $-x^* \in S^*(\varepsilon)$. ■

5. CONCLUDING REMARKS

In this section we would like to summarize briefly the developments given in the preceding sections. In Section 2 we provide sufficient conditions for various (standard) convexity properties of the optimal value function f^* . The notion of convexity of R and the basic result in Proposition 2.2 (for R convex) are well known. We slightly extend the definition of convexity by introducing essential convexity and closure convexity and generalize this basic result in Propositions 2.2 and 2.4. Proposition 2.6 gives general sufficient conditions for convexity of R for the parametric NLP problem $P_3(\varepsilon)$ which enables us to generalize the result of Mangasarian and Rosen (1964) (Corollary 2.7) and forms the basis for applications of most other results to $P_3(\varepsilon)$. Specialization of this proposition to the grhs NLP problem $P_2(\varepsilon)$ is standard. A partial extension of Proposition 2.2 (see Proposition 2.10) is obtained using a new notion of hull convexity of R . We introduce the notion of convexity at a point for R and obtain another extension of the basic result in Propositions 2.11 and 2.12. Results concerning strict convexity of f^* (Propositions 2.13 - 2.15) and uniform convexity of f^* (Proposition 2.17) appear to be new. Similar results on polyhedral convexity (Propositions 2.19 - 2.20) and positive homogeneity (Propositions 2.21 - 2.23) were obtained previously in a different setting by Rockafellar (1970) and Borwein (1980), respectively. Results on homogeneous convexity (Propositions 2.24 - 2.25) were obtained earlier in more special cases.

Concavity properties of f^* are considered in Section 3. Unlike in Section 2, virtually all the results in this section (except for

Proposition 3.9) appear to be new at this level of generality. The notion of concavity of R , although known, has not been widely used and the basic result in Proposition 3.1 seems to be new in the literature. By introducing the notion of hull concavity of R we are able to further extend this basic result in Proposition 3.4. Similar to the convex case, the notions of closure concavity and weak closure concavity allow for a slight extension of Proposition 3.1 (see Proposition 3.6). It is difficult, however, to specialize these results to the parametric NLP problem $P_3(\varepsilon)$ and grhs NLP problem $P_2(\varepsilon)$. One such application is given in Proposition 3.8. Undoubtedly, other results of that type would be of interest. Proposition 3.9 is a special result which is apparently well known. Proposition 3.10 is closely related and appears to be new. An interesting novel application of Proposition 3.9 is given in Proposition 3.11.

Introduction of the notion of concavity and hull concavity at a point for R leads to an extension of basic Propositions 3.1 and 3.4 (Proposition 3.13). Proposition 3.9 can be similarly extended (see Proposition 3.14). Results on strict concavity (Propositions 3.15 - 3.17) and the notion of strict concavity of R appear to be new, as do the results on uniform concavity (Propositions 3.18 - 3.19) and homogeneous concavity (Propositions 3.20 - 3.21).

Section 4 contains results concerning both the optimal value function f^* and the solution set map S^* . Proposition 4.2 is the basic result of this section and appears to be new. It demonstrates the surprising fact that essentially the same conditions which imply convexity of S^* as a point-to-set map also imply concavity, and hence, affineness of S^*. Affineness of f^* (in the case when f^* is finite) easily follows from the basic results of Sections 2 and 3 and their extended versions in the case of a hull affine point-to-set map R. Strengthening the assumptions of Proposition 4.2 gives a stronger result in Corollary 4.4. By introducing the notion of a closure affine map R we obtain affineness of f^* (for finite f^*) under slightly different assumptions in Proposition 4.5. Extension of Proposition 4.2 is obtained using the notion of a point-to-set map affine at a point (see Proposition 4.6). Finally, two results are obtained giving sufficient conditions for linearity of f^* and homogeneous affineness or linearity of S^* (Propositions 4.7 and 4.9).

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